

Some Characteristics of Mahalanobis Distance for Bivariate Probability Distributions

Anwar H. Joarder

Received January 14, 2008; Revised May 18, 2008

Abstract

The standardized moments or Mahalanobis moments are easily calculated for bivariate elliptical distribution which includes bivariate normal and bivariate t -distributions as special cases. But these are challenging for many bivariate distributions. A set of alternative formulae is developed to derive Mahalanobis moments for any bivariate distribution. The second order Mahalanobis moment accounts for the coefficient of kurtosis. Ideas are illustrated with examples.

Keywords: Bivariate distribution, standardized moments, Mahalanobis distance, product moments, kurtosis.

2000 Mathematics Subject Classification: 60E10, 60E05, 62E10.

1 Introduction

In a series of papers, Mardia [13-17] defined and discussed the properties of measures of kurtosis and skewness based on Mahalanobis distance. Interested readers may go through Kotz *et al.* [10] for an excellent discussion on multivariate skewness and kurtosis. The coefficient of kurtosis is the second order moment of standardized distance better known as Mahalanobis distance. The moments may be called standardized moments, Mahalanobis moments or Mardia moments.

For some distributions, it is easy to derive the distribution of the Mahalanobis distance and calculate moments, but for others it is difficult. Kotz *et al.* [10] have provided the coefficient of kurtosis of the Marshall-Olkin bivariate exponential distribution. In this paper, we provide an alternative method to calculate Mahalanobis moments for bivariate distributions in terms of the product moments of the components of a bivariate vector. Product moments (also called raw product moments or product moments around zero) of order a and b for

The author gratefully acknowledges the excellent research support provided by King Fahd University of Petroleum and Minerals, Saudi Arabia through the Project FT 2004-22.

two random variables X and Y are defined by $\mu(a, b) = E(X^a Y^b)$ while the centered product moments (sometimes called central product moments, corrected product moments or central mixed moments) are defined by

$$\mu(a, b) = E[(X - E(X))^a (Y - E(Y))^b]. \quad (1.1)$$

Interested readers may go through Johnson *et al.* [6, p.3] or Johnson *et al.* [7, p.46]. Evidently $\mu'(a, 0) = E(X^a)$ is the a -th moment of X , and $\mu'(0, b) = E(Y^b)$ is the b -th moment of Y . In case X and Y are independent, then $\mu'(a, b) = E(X^a)E(Y^b) = \mu'(a, 0)\mu'(0, b)$ and $\mu(a, b) = \mu(a, 0)\mu(0, b)$. The correlation coefficient ρ ($-1 < \rho < 1$) between X and Y is denoted by

$$\rho_{X,Y} = \frac{\mu(1, 1)}{\sqrt{\mu(2, 0)\mu(0, 2)}}. \quad (1.2)$$

Note that $\mu(2, 0) = E(X - E(X))^2 = V(X) = \sigma_{20}$ while the central product moment, $\mu(1, 1) = E[(X - E(X))(Y - E(Y))]$ is, in fact, the covariance between X and Y .

This paper is based on Joarder [4]. In this paper, we derive Mahalanobis moments in terms of centered product moments in Section 2. In particular, Mahalanobis moments for bivariate normal distribution and bivariate t -distribution are derived in Section 4 by the method developed in Section 2. It is observed that general formulae for Mahalanobis moments for bivariate elliptical distribution, which include bivariate normal and t -distributions as special cases, are easily obtained. An example of bivariate chi-square distribution is mentioned for which the proposed method developed in Section 2 is appropriate. In what follows we denote $\mu(a, b) = E[(X - \xi)^a (Y - \theta)^b]$ where $\xi = E(X)$, $\theta = E(Y)$.

2 Mahalanobis Moments in Terms of Product Moments

For a bivariate random vector $W = (X, Y)'$, with mean vector $\mu = (\xi, \theta)'$ and covariance matrix

$$Cov(W) = E[(W - \mu)(W - \mu)'] = \begin{pmatrix} \mu(2, 0) & \mu(1, 1) \\ \mu(1, 1) & \mu(0, 2) \end{pmatrix} = \Omega \text{ (say),}$$

the standardized distance is defined by

$$\begin{aligned} Q &= (W - \mu)' \Omega^{-1} (W - \mu) \\ &= ((X - \xi), (Y - \theta)) \Omega^{-1} ((X - \xi), (Y - \theta))'. \end{aligned} \quad (2.1)$$

The quantity Q is also known as generalized distance or Mahalanobis distance. For a bivariate random vector W with $E(W) = \mu$ and $Cov(W) = \Omega$, we define standardized moments or Mahalanobis moments by $\beta_i = E(Q^i)$, $i = 1, 2, \dots$, where $Q = (W -$

$\mu)' \Omega^{-1}(W - \mu) = \|\Omega^{-1/2}(W - \mu)\|^2$. Of special interest is the second order Mahalanobis moments by $\beta_2 = E(Q^2)$ which is the coefficient of kurtosis in the sense of Mardia [15]. For a recent paper on kurtosis, see An and Ahmed [1], and the references therein.

Note that Nadarajah and Mitov [18] presented an elegant technique for product moments of the components of any multivariate random vector in terms of cumulative distribution function or survival function. It appears that if the cumulative distribution function or the survival function has a closed form, the Nadarajah and Mitov [18] technique works well. For Marshall-Olkin bivariate exponential distribution with survival function

$$P(X \geq x, Y \geq y) = \begin{cases} e^{-x-(1+\lambda)y}, & 0 \leq x \leq y \\ e^{-y-(1+\lambda)x}, & 0 \leq y \leq x, \end{cases}$$

where $\lambda > 0$, Nadarajah and Mitov [18] calculated raw product moment of general order from which it is possible to calculate Mahalanobis moments of the distribution. Kotz *et al.* [10, p.82] mentioned that the coefficient of kurtosis of the distribution is given by

$$\beta_2 = 2(1 + \rho)^{-3}(3\rho^4 + 9\rho^3 + 15\rho^2 + 12\rho + 4),$$

where the correlation coefficient ρ is given by $(\lambda + 2)\rho = \lambda$. They also mentioned that in case $\rho = 0$, the components X and Y become independent, in which case $\beta_2 = 8$ (which is the same as that of the bivariate normal distribution). Interested readers may go through Kotz *et al.* [12] for a useful formula for product moments for any univariate distribution.

In fact, regardless of the distribution of the variable in question the first order standardized moment is the dimension of the random variable. Let W be a p -component random vector with $E(W) = \mu$ and $Cov(W) = \Omega$, then

$$\begin{aligned} \beta_1 &= E[tr(W - E(W))' \Omega^{-1}(W - E(W))] \\ &= E(tr \Omega^{-1}(W - E(W))(W - E(W))') \\ &= tr[\Omega^{-1}E(W - E(W))(W - E(W))'] \\ &= tr(\Omega^{-1}\Omega) \\ &= p. \end{aligned} \tag{2.2}$$

If $W \sim N_p(\mu, \Omega)$, then $Q = (W - \mu)' \Omega^{-1}(W - \mu) \sim \chi_p^2$ so that $\beta_k = E(Q^k)$ and the standardized moment of order k follows from the chi-square distribution. In case W has a multivariate t -distribution or multivariate elliptical distribution, standardized moments can also be calculated without much difficulty. In this paper, we develop general formulae for standardized moments for any bivariate distribution. We also provided some examples to illustrate the idea. We remark that in case W has a complicated distribution, say a bivariate chi-square distribution (Theorem 4.4), it would be much difficult to calculate the distribution of Q and hence the standardized moments. In the following theorem, we derive a set of formulae for standardized moments for any bivariate distribution in terms of centered product moments just to demonstrate the potential of an alternative way.

Theorem 2.1. Let $\mu(a, b)$ be the centered product moments between X and Y as defined by (1.1), and Q be the Mahalanobis distance as defined by (2.1). Then

$$\begin{aligned}
 (i) \quad & [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]^2 E(Q^2) \\
 &= \mu(4, 0)\mu^2(0, 2) + \mu(0, 4)\mu^2(2, 0) + 4\mu^2(1, 1)\mu(2, 2) + 2\mu(2, 0)\mu(0, 2)\mu(2, 2) \\
 &\quad - 4\mu(0, 2)\mu(1, 1)\mu(3, 1) - 4\mu(2, 0)\mu(1, 1)\mu(1, 3), \\
 (ii) \quad & [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]^3 E(Q^3) \\
 &= \mu(6, 0)\mu^3(0, 2) + \mu^3(2, 0)\mu(0, 6) - 6\mu^2(0, 2)\mu(1, 1)\mu(5, 1) \\
 &\quad - 6\mu^2(2, 0)\mu(1, 1)\mu(1, 5) + 12\mu(0, 2)\mu^2(1, 1)\mu(4, 2) + 12\mu(2, 0)\mu^2(1, 1)\mu(2, 4) \\
 &\quad + 3\mu(2, 0)\mu^2(0, 2)\mu(4, 2) + 3\mu^2(2, 0)\mu(0, 2)\mu(2, 4) \\
 &\quad - 8\mu^3(1, 1)\mu(3, 3) - 12\mu(2, 0)\mu(0, 2)\mu(1, 1)\mu(3, 3).
 \end{aligned}$$

Proof. From (2.1), we have

$$Q = (X - \xi, Y - \theta) \begin{pmatrix} \mu(2, 0) & \mu(1, 1) \\ \mu(1, 1) & \mu(0, 2) \end{pmatrix}^{-1} \begin{pmatrix} X - \xi \\ Y - \theta \end{pmatrix}, \quad (2.3)$$

which can be simplified to

$$\begin{aligned}
 [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]Q &= \mu(0, 2)(X - \xi)^2 - 2\mu(1, 1)(X - \xi)(Y - \theta) \\
 &\quad + \mu(2, 0)(Y - \theta)^2.
 \end{aligned} \quad (2.4)$$

By taking expected values on both sides of the above identity, we find that

$$\begin{aligned}
 [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]E(Q) &= \mu(2, 0)\mu(0, 2) - 2\mu^2(1, 1) + \mu(0, 2)\mu(2, 0) \\
 &= 2[\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)].
 \end{aligned}$$

That is, $E(Q) = 2$, which is the dimension of the vector $W = (X, Y)'$. The notion is generally true, see (2.2). By squaring both sides of (2.4), we get

$$\begin{aligned}
 [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]^2 Q^2 &= \mu^2(0, 2)(X - \xi)^4 + 4\mu^2(1, 1)(X - \xi)^2(Y - \theta)^2 \\
 &\quad + \mu^2(2, 0)(Y - \theta)^4 - 4\mu(0, 2)\mu(1, 1)(X - \xi)^3(Y - \theta) \\
 &\quad + 2\mu(2, 0)\mu(0, 2)(X - \xi)^2(Y - \theta)^2 \\
 &\quad - 4\mu(2, 0)\mu(1, 1)(X - \xi)(Y - \theta)^3.
 \end{aligned}$$

Then the result in (i) follows by taking expected values on both sides of the above identity. By cubing both sides of (2.4), we have

$$\begin{aligned}
 & [\mu(2, 0)\mu(0, 2) - \mu^2(1, 1)]^3 Q^3 \\
 &= \mu^3(0, 2)(X - \xi)^6 - 8\mu^3(1, 1)(X - \xi)^3(Y - \theta)^3 + \mu^3(2, 0)(Y - \theta)^6
 \end{aligned}$$

$$\begin{aligned}
& -6\mu^2(0,2)\mu(1,1)(X-\xi)^5(Y-\theta) + 12\mu(0,2)\mu^2(1,1)(X-\xi)^4(Y-\theta)^2 \\
& + 3\mu(2,0)\mu^2(0,2)(X-\xi)^4(Y-\theta)^2 + 3\mu^2(2,0)\mu(0,2)(X-\xi)^2(Y-\theta)^4 \\
& + 12\mu(2,0)\mu^2(1,1)(X-\xi)^2(Y-\theta)^2 - 6\mu^2(2,0)\mu(1,1)(X-\xi)(Y-\theta)^5 \\
& - 12\mu(2,0)\mu(0,2)\mu(1,1)(X-\xi)^3(Y-\theta)^3.
\end{aligned}$$

Part (ii) follows by taking expected values of the above identity. $\square$

Corollary 2.1. Let $\mu(a, b)$ be the centered product moment defined by (1.1), ρ be the coefficient of correlation defined by (1.2) and Q be the Mahalanobis distance defined by (2.1). Then we have the following:

$$\begin{aligned}
\text{(i)} \quad & [\mu(2,0)\mu(0,2)(1-\rho^2)]^2 E(Q^2) \\
& = \mu(4,0)\mu^2(0,2) + \mu(0,4)\mu^2(2,0) + (4\rho^2 + 2)\mu(2,0)\mu(0,2)\mu(2,2) \\
& \quad - 4\rho(\mu(2,0)\mu(0,2))^{1/2}[\mu(0,2)\mu(3,1) + \mu(2,0)\mu(1,3)], \\
\text{(ii)} \quad & [\mu(2,0)\mu(0,2)(1-\rho^2)]^3 E(Q^3) \\
& = \mu(6,0)\mu^3(0,2) + \mu^3(2,0)\mu(0,6) - 4(\mu(2,0)\mu(0,2))^{3/2}\mu(3,3)\rho(2\rho^2 + 3) \\
& \quad - 6(\mu(2,0)\mu(0,2))^{1/2}\rho[\mu^2(0,2)\mu(5,1) + \mu^2(2,0)\mu(1,5)] \\
& \quad + 3\mu(2,0)\mu(0,2)(4\rho^2 + 1)[\mu(0,2)\mu(4,2) + \mu(2,0)\mu(2,4)].
\end{aligned}$$

Corollary 2.2. Let X and Y have a bivariate distribution with centered product moments $\mu(a, b)$ defined by (1.1), correlation coefficient ρ defined by (1.2) and Q defined by (2.1). Further let $\mu(a, b) = \mu(b, a)$. Then we have the following:

$$\begin{aligned}
\text{(i)} \quad & \mu^2(0,2)(1-\rho^2)^2 E(Q^2) = 2\mu(4,0) + (4\rho^2 + 2)\mu(2,2) - 8\rho\mu(3,1), \\
\text{(ii)} \quad & \mu^3(0,2)(1-\rho^2)^3 E(Q^3) = 2\mu(6,0) - (8\rho^3 + 12\rho)\mu(3,3) - 12\rho\mu(5,1) \\
& \quad + (24\rho^2 + 6)\mu(4,2).
\end{aligned}$$

Corollary 2.3. Let X and Y have a bivariate distribution with product moments $\mu(a, b)$ defined by (1.1) and Q defined by (2.1). If X and Y are independent, then we have the following:

$$\begin{aligned}
\text{(i)} \quad & E(Q^2) = 2 + \frac{\mu(4,0)}{\mu^2(2,0)} + \frac{\mu(0,4)}{\mu^2(0,2)}, \\
\text{(ii)} \quad & E(Q^3) = \frac{\mu(6,0)}{\mu^3(2,0)} + \frac{\mu(0,6)}{\mu^3(0,2)} + 3 \left(\frac{\mu(4,0)}{\mu^2(2,0)} + \frac{\mu(0,4)}{\mu^2(0,2)} \right).
\end{aligned}$$

Corollary 2.4. Let X and Y have a bivariate distribution with product moments $\mu(a, b)$ defined by (1.1) and Q defined by (2.1). If X and Y are independently and identically distributed, then we have the following:

$$\text{(i)} \quad E(Q^2) = 2 \left(1 + \frac{\mu(4,0)}{\mu^2(2,0)} \right),$$

$$(ii) \quad E(Q^3) = 2 \left(1 + \frac{\mu(6,0)}{\mu^3(2,0)} + 3 \frac{\mu(4,0)}{\mu^2(2,0)} \right).$$

3 Centered Product Moments of the Bivariate Normal Distribution

The pdf (probability density function) of the bivariate normal distribution is given by

$$f_1(x, y) = \frac{(1 - \rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2} \exp \left(\frac{-q(x, y)}{2} \right), \quad (3.1)$$

where

$$(1 - \rho^2)q(x, y) = \left(\frac{x - \xi}{\sigma_1} \right)^2 + \left(\frac{y - \theta}{\sigma_2} \right)^2 - \frac{2\rho(x - \xi)(y - \theta)}{\sigma_1\sigma_2}.$$

The following theorem is due to Kendal and Stuart [8, p.91].

Theorem 3.1. *The centered product moments $\mu(a, b) = E[(X - \xi)^a(Y - \theta)^b]$ of the bivariate normal distribution with pdf in (3.1) are given by*

$$\begin{aligned} \mu(a, b) &= \sigma_1^a \sigma_2^b \lambda(a, b), \\ \lambda(a, b) &= (a+b-1)\rho\lambda(a-1, b-1) + (a-1)(b-1)(1-\rho^2)\lambda(a-2, b-2), \\ \lambda(2a, 2b) &= \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}, \\ \lambda(2a+1, 2b+1) &= \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}, \\ \lambda(2a, 2b+1) &= \lambda(2a+1, 2b) = 0. \end{aligned}$$

The above can be rewritten as

$$\begin{aligned} \mu(a, b) &= (a+b-1)\rho\sigma_1\sigma_2\mu(a-1, b-1) \\ &\quad + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2, b-2), \\ \mu(2a, 2b) &= \sigma_1^{2a}\sigma_2^{2b} \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}, \\ \mu(2a+1, 2b+1) &= \sigma_1^{2a+1}\sigma_2^{2b+1} \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}, \\ \mu(2a, 2b+1) &= \mu(2a+1, 2b) = 0. \end{aligned}$$

Product moments of bivariate normal distribution that are needed for deriving standardized moments up to order 3 are provided below:

$$\mu(2, 0) = \sigma_1^2, \quad \mu(0, 2) = \sigma_2^2, \quad \mu(4, 0) = 3\sigma_1^4, \quad \mu(0, 4) = 3\sigma_2^4, \quad \mu(6, 0) = 15\sigma_1^6,$$

$$\begin{aligned}\mu(0, 6) &= 3\sigma_2^6, \mu(1, 1) = \rho\sigma_1\sigma_2, \mu(2, 2) = (1 + 2\rho^2)\sigma_1^2\sigma_2^2, \mu(1, 3) = 3\rho\sigma_1\sigma_2^3, \\ \mu(3, 1) &= 3\rho\sigma_1^3\sigma_2, \mu(2, 4) = 3(1 + 4\rho^2)\sigma_1^2\sigma_2^4, \mu(3, 3) = 3\rho(3 + 2\rho^2)\sigma_1^3\sigma_2^3, \\ \mu(4, 2) &= 3(1 + 4\rho^2)\sigma_1^4\sigma_2^2, \mu(1, 5) = 15\rho\sigma_1\sigma_2^5, \mu(5, 1) = 15\rho\sigma_1^5\sigma_2.\end{aligned}$$

4 Some Examples

4.1 Bivariate normal distribution

Let us represent the bivariate normal distribution with pdf in (3.1) by

$$W = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \Sigma), \quad \mu = \begin{pmatrix} \xi \\ \theta \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \mu(2, 0) & \mu(1, 1) \\ \mu(1, 1) & \mu(0, 2) \end{pmatrix}.$$

It is known that for a p -variate normal distribution, $W \sim N_p(\mu, \Sigma)$, the standardized distance $Q = (W - \mu)' \Sigma^{-1} (W - \mu) \sim \chi_p^2$ so that $\beta_1 = E(Q) = p$, $\beta_2 = E(Q^2) = p(p+2)$ and $\beta_3 = E(Q^3) = p(p+2)(p+4)$. In particular, for the univariate normal distribution, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 15$ and for the bivariate normal distribution,

$$\beta_1 = 2, \quad \beta_2 = 8, \quad \beta_3 = 48. \quad (4.1)$$

We derive Mahalanobis moments for bivariate normal in Theorem 4.2, and bivariate t -distribution in Theorem 4.3 by the method developed in Section 2. For bivariate elliptical distribution, Mahalanobis moments are known in terms of the moments of radius. However, these are difficult for many other distributions as it is difficult to derive the distribution of the standardized distances of these distributions.

Theorem 4.1. *The second and the third order standardized moments of bivariate normal distribution are given by $\beta_2 = 8$ and $\beta_3 = 48$, respectively.*

Proof. By the use of moments from Section 3, it follows from Theorem 2.1(i) or preferably from Corollary 2.1(i) that

$$\begin{aligned}[\sigma_1^2\sigma_2^2(1 - \rho^2)]^2 E(Q^2) &= (3\sigma_1^4)(\sigma_2^2)^2 + (3\sigma_2^4)(\sigma_1^2)^2 + 4(\rho^2\sigma_1^2\sigma_2^2)[\sigma_1^2\sigma_2^2(1 + 2\rho^2)] \\ &\quad - 4(\sigma_2^2)(\rho\sigma_1\sigma_2)(3\sigma_1^3\sigma_2\rho) - 4\sigma_1^2(\rho\sigma_1\sigma_2)(3\sigma_1\sigma_2^3\rho) \\ &\quad + 2\sigma_1^2\sigma_2^2[\sigma_1^2\sigma_2^2(1 + 2\rho^2)].\end{aligned}$$

So

$$\begin{aligned}(1 - \rho^2)^2 E(Q^2) &= 3 + 3 + 4\rho^2(1 + 2\rho^2) + 2(1 + 2\rho^2) - 4\rho(3\rho) - 4\rho(3\rho) \\ &= 8(1 - 2\rho^2 + \rho^4).\end{aligned}$$

Similarly by plugging in the moments from Section 3, it follows from Theorem 2.1(ii) or preferably from Corollary 2.1(ii) that

$$[\sigma_1^2\sigma_2^2(1 - \rho^2)]^3 E(Q^3) = (15\sigma_1^6)(\sigma_2^2)^3 + (\sigma_1^2)^3(15\sigma_2^6) - 8(\rho^3\sigma_1^3\sigma_2^3)[3\sigma_1^3\sigma_2^3\rho(3 + 2\rho^2)]$$

$$\begin{aligned}
& -12\sigma_1^2\sigma_2^2(\rho\sigma_1\sigma_2)[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)] - 6\sigma_2^4(\rho\sigma_1\sigma_2)(15\sigma_1^5\sigma_2\rho) \\
& - 6\sigma_1^4(\rho\sigma_1\sigma_2)(15\sigma_1\sigma_2^5\rho) + 12\sigma_2^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^4\sigma_2^2(1+4\rho^2)] \\
& + 12\sigma_1^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)] + 3\sigma_1^2(\sigma_2^2)^2[3\sigma_1^4\sigma_2^2(1+4\rho^2)] \\
& + 3(\sigma_1^2)^2(\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)].
\end{aligned}$$

So

$$\begin{aligned}
(1-\rho^2)^3 E(Q^3) &= 15 + 15 - 8\rho^3[3\rho(3+2\rho^2)] - 12\rho(3\rho(3+2\rho^2)) \\
& - 6\rho(15\rho) - 6\rho(15\rho) + 12\rho^2[3(1+4\rho^2)] \\
& + 3[3(1+4\rho^2)] + 3[3(1+4\rho^2)] \\
& = 48(1-3\rho^2+3\rho^4-\rho^6). \quad \square
\end{aligned}$$

4.2 Bivariate t -distribution

Let $X' = (X_1, X_2)$ be the bivariate t -random vector with pdf

$$f_2(x) = (2\pi)^{-1} |\Sigma|^{1/2} \left(1 + (x - \theta)' (\nu\Sigma)^{-1} (x - \theta) \right)^{-(\nu+2)/2}, \quad (4.2)$$

where $\theta' = (\theta_1, \theta_2)$ is an unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters while the scalar ν is assumed to be a known positive constant [2, p.289]. For recent update on t -distributions, see Kotz and Nadarajah [11] and Kibria [9] and the references therein.

The following theorem, due to Joarder [3], is needed to calculate Mahalanobis moments of the above bivariate t -distribution given by (4.2).

Theorem 4.2. *The centered product moments of the bivariate t -distribution with pdf in (4.1) are given by*

$$\begin{aligned}
\mu(a, b; \nu) &= (a+b-1)\rho\sigma_1\sigma_2\mu(a-1, b-1)\gamma_2 \\
& + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2, b-2)\gamma_4, \\
\mu(2a, 2b; \nu) &= \sigma_1^{2a}\sigma_2^{2b} \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}\gamma_{2a+2b}}{(a-j)!(b-j)!(2j)!}, \\
\mu((2a+1, 2b+1; \nu) &= \sigma_1^{2a+1}\sigma_2^{2b+1} \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}\gamma_{2a+2b+2}}{(a-j)!(b-j)!(2j+1)!}, \\
\mu(2a, 2b+1; \nu) &= \mu(2a+1, 2b; \nu) = 0
\end{aligned}$$

where

$$\gamma_a = \frac{(\nu/2)^{a/2}\Gamma((\nu-a)/2)}{\Gamma(\nu/2)}, \quad \nu > a.$$

By the use of the above moments in Theorem 2.1 or preferably in Corollary 2.1, we find the second and third order Mahalanobis moments of the bivariate t -distribution having pdf in (4.1)

$$\beta_2 = \frac{8(\nu - 2)}{\nu - 4}, \quad \nu > 4,$$

$$\beta_3 = \frac{48(\nu - 2)^2}{(\nu - 4)(\nu - 6)}, \quad \nu > 6.$$

Notice that as $\nu \rightarrow \infty$, the above moments converges to $\beta_2 = 8$ and $\beta_3 = 48$, as known for bivariate normal distribution.

4.3 Multivariate elliptical distribution

The second and third order Mahalanobis moments are calculated for p -variate elliptical distribution. With $p = 2$, the results boil down to bivariate elliptical distribution. Consider the multivariate elliptical distribution of X with pdf

$$f_3(x) = g((x - \mu)' \Sigma^{-1}(x - \mu)), \quad (4.3)$$

where x is a p -dimensional column vector with mean $E(X) = \mu$ and the covariance matrix $Cov(X) = p^{-1}E(R^2)\Sigma$ where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X - \mu)$. Then we have the following theorem (cf. Anderson [2, p.103]):

Theorem 4.3. *Let X have the multivariate elliptical distribution with pdf in (4.3). Then the second and the third order Mahalanobis moments of the distribution are given by*

$$\beta_2 = p^2 \frac{E(R^4)}{E^2(R^2)} \quad \text{and} \quad \beta_3 = p^3 \frac{E(R^6)}{E^3(R^2)},$$

respectively, where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X - \mu)$.

Proof. The covariance matrix of the elliptical distribution is given by $Cov(X) = p^{-1}E(R^2)\Sigma$ so that the standardized distance is given by

$$Q = (X - \mu)'(p^{-1}E(R^2)\Sigma)^{-1}(X - \mu).$$

Then the theorem is obvious by virtue of

$$Q = \frac{pR^2}{E(R^2)}, \quad Q^2 = \frac{p^2R^4}{E^2(R^2)}, \quad Q^3 = \frac{p^3R^6}{E^3(R^2)}. \quad \square$$

Note that if the form of $g(\cdot)$ is known, the second and the third order Mahalanobis moments of the distribution can be calculated by the pdf of R given by

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} g(r^2), \quad 0 < r.$$

It is well known that for the multivariate normal distribution $R^2 \sim \chi_p^2$, and for the multivariate t -distribution with pdf

$$f_4(x) = \frac{\Gamma((\nu+p)/2)}{\Gamma(\nu/2)(\nu\pi)^{p/2}} |\Sigma|^{-1/2} \left(1 + (x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-(\nu+p)/2}, \quad \nu > 2,$$

we have $p^{-1}R^2 \sim F(p, \nu)$. We remark that β_2 and β_3 can be directly calculated by Theorem 2.1.

4.4 Bivariate chi-square distribution

The following bivariate chi-square distribution is derived by Joarder [5].

Theorem 4.4. *The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its pdf is given by*

$$f_5(u, \nu) = \frac{(u\nu)^{(m-2)/2} \exp(-(u+\nu)/(2-2\rho^2))}{2^m \sqrt{\pi} \Gamma(m/2) (1-\rho^2)^{m/2}} \\ \times \sum_{k=0}^{\infty} [1 + (-1)^k] \left(\frac{\rho\sqrt{u\nu}}{1-\rho^2} \right)^k \frac{\Gamma((k+1)/2)}{k! \Gamma((k+m)/2)},$$

where $m > 2, -1 < \rho < 1$.

Since it is difficult to derive the distribution of Mahalanobis distance of the above bivariate chi-square distribution, Mahalanobis moments of the above bivariate chi-square distribution can be derived by using the results developed in Section 2. In case $\rho = 0$, the pdf of the joint probability distribution in Theorem 4.4, would be that of the product of two independent chi-square random variables $U \sim \chi_m^2$ and $V \sim \chi_m^2$.

References

- [1] L. An and S. E. Ahmed, *Improving the performance of kurtosis parameter*, Comput. Statist. Data Anal., to appear.
- [2] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, John Wiley, New York, 2003.
- [3] A. H. Joarder, *On some characteristics of the bivariate t -distribution*, Int. J. Mod. Math. **2** (2007), 191–204.
- [4] A. H. Joarder, *Mahalanobis moments of bivariate Distributions*, in: Festschrift in Honor of George and Frances Ball Distinguished Professor of Statistics 'Mir Masoom Ali', (Ball State University, Indiana, U.S.A.), 2007, pp. 201–206.
- [5] A. H. Joarder, *Moments of product and ratio of two chi-square variables*, Statist. Papers, to appear.

- [6] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Discrete Multivariate Distributions*, John Wiley, 1997.
- [7] N. L. Johnson, S. Kotz, and A. W. Kemp, *Univariate Discrete Distributions*, John Wiley, 1993.
- [8] M. G. Kendal and A. Stuart, *The Advanced Theory of Statistics*, Charles Griffin and Co. Ltd., London, 1969.
- [9] B. M. G. Kibria, *The matrix t -distributions and its applications in predictive inference*, J. Mult. Anal. **97** (2006), 785–795.
- [10] S. Kotz, N. Balakrishnan, and N. L. Johnson, *Continuous Multivariate Distributions*, John Wiley and Sons, New York, 2000.
- [11] S. Kotz and S. Nadarajah, *Multivariate t -Distributions and Their Applications*, Cambridge University Press, 2005.
- [12] S. Kotz, S. Nadarajah, and K. V. Mitov, *A useful formula for product moments*, Student **4** (2003), 239–244.
- [13] K. V. Mardia, *A translation family of bivariate distributions and Frechet's bounds*, Sankhya **32A** (1970), 119–121.
- [14] K. V. Mardia, *Families of Bivariate Distributions*, Griffin, London, 1970.
- [15] K. V. Mardia, *Measure of multivariate skewness and kurtosis with applications*, Biometrika **57** (1970), 519–530.
- [16] K. V. Mardia, *Applications of some measures of multivariate skewness and kurtosis for testing normality and robustness studies*, Sankhya **36B** (1974), 115–128.
- [17] K. V. Mardia, *Assessment of multinormality and the robustness of the Hotelling's T^2 test*, Appl. Statist. **24** (1975), 163–171.
- [18] S. Nadarajah and K. Mitov, *Product moments of multivariate random vectors*, Comm. Statist. Theory Methods **32** (2003), 47–60.

Anwar H. Joarder: Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia.

E-mail address: anwarj@kfupm.edu.sa