

ON THE CHARACTERISTIC FUNCTION OF THE MULTIVARIATE t -DISTRIBUTION

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Abstract

This paper derives the characteristic function of the multivariate t -distribution in terms of a well-known special function, namely, the Macdonald function, which has been extensively studied by several authors in recent years.

Key Words

t -distribution, characteristic function of t -distribution, multivariate t -distribution, Macdonald function.

1. INTRODUCTION

The multivariate t -distribution, a natural generalization of the univariate Student t -distribution, has been studied in various contexts by several authors. Among them we mention Bechhofer, Dunnet and Sobel (1954), Cornish (1954), Sutradhar and Ali (1986) and Sutradhar (1990).

The characteristic function of the univariate Student t -distribution for odd degrees of freedom has been derived by Fisher and Healy (1956). Ifram (1970) gives a general result for all degrees of freedom but Pestana (1977) points out that this result is incorrect. A series representation of the characteristic function of the multivariate t -distribution has been given by Sutradhar (1986). However, the convergence of those series has not been ever checked. Javier and Śrivastava (1988) also find a series representation for the characteristic function of the multivariate t -distribution. We find the characteristic function in terms of the well-known Macdonald function.

In particular, we consider a p -variate random variable $X = (X_1, X_2, \dots, X_p)'$ having probability density function

$$f(x) = C(\nu, p) |\nu\Lambda|^{-1/2} (1 + (x - \theta)'(\nu\Lambda)^{-1}(x - \theta))^{-(\nu+p)/2}, \quad (1.1)$$

where the normalizing constant $C(\nu, p)$ is given by

$$C(\nu, p) = \Gamma((\nu + p)/2) / (\pi^{p/2} \Gamma(\nu/2)). \quad (1.2)$$

The density (1.1) is known as the p -variate t -distribution where the p -dimensional column vector θ is the location parameter, the $p \times p$ matrix Λ is the positive definite matrix of scale parameters while ν represents the degrees of freedom.

An elegant expression for the characteristic function of the multivariate t -distribution in terms of Macdonald function has been derived in this paper.

2. THE MACDONALD FUNCTION

The Macdonald function has been getting increasing attention in the recent statistical literature (see for example, Barndorff-Nielsen and Blaesild, 1981). We list below some standard results on the integral and series representation of the Macdonald function which will be required in the sequel. In relation to the present work the Macdonald function $K_\alpha(r)$ with argument r and order α may be defined by the following integral representation (see for example, Watson, 1958, p 172):

$$K_\alpha(r) = \left(\frac{2}{r}\right)^\alpha \frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}} \int_0^\infty (1 + u^2)^{-(\alpha+1/2)} \cos ru \, du, \quad r > 0, \alpha > -1/2 \quad (2.1)$$

In particular, we remark that (see, for example, Tranter 1968, p 19)

$$K_{1/2}(r) = \sqrt{\pi}(2r)^{-1/2} e^{-r}. \quad (2.2)$$

A series representation of the Macdonald function $K_\alpha(r)$ where $r > 0$ and α , a nonnegative integer is as follows (see, for example, Lebedev, 1965, pp 107, 110):

$$\begin{aligned} K_\alpha(r) &= 2^{\alpha-1} \sum_{j=0}^{\alpha-1} \frac{(-1)^j (\alpha - 1 - j)!}{j! 4^j} r^{2j-\alpha} \\ &+ (-1)^\alpha 2^{-(\alpha+1)} \sum_{j=0}^{\infty} \frac{\{\xi(1+j) + \xi(\alpha+1+j) - \ln(r^2/4)\}}{j! (\alpha+j)! 4^j} r^{2j+\alpha} \quad (2.3) \\ &\text{with } \xi(r) = \Gamma'(r)/\Gamma(r). \quad (2.4) \end{aligned}$$

For non-integral positive values of α , a series representation of $K_\alpha(r)$ for $r > 0$ is given by (see, for example, Spainer and Oldham, 1987, chapter 51):

$$K_\alpha(r) = 2^{\alpha-1} \sum_{j=0}^{\infty} \frac{r^{2j-\alpha}}{j! (1-\alpha)_j 4^j} + 2^{-(\alpha+1)} \sum_{j=0}^{\infty} \frac{r^{2j+\alpha}}{j! (1+\alpha)_j 4^j} \quad (2.5)$$

where $(\alpha)_j$ is the Pochhammer polynomial defined by

$$(\alpha)_j = \Gamma(\alpha + j)/\Gamma(\alpha) = \alpha(\alpha + 1) \dots (\alpha + j - 1). \quad (2.6)$$

3. THE CHARACTERISTIC FUNCTION

The main result of the paper is presented below in the form of a theorem.

Theorem 3.1 Let X have the multivariate t -distribution as defined by (1.1). Then the characteristic function $\phi_X(t) = E[\exp(it'X)]$ of X is given by

$$\phi_X(t) = \exp(it'\theta) \frac{\|(\nu\Lambda)^{1/2}t\|^{\nu/2}}{2^{\nu/2-1}\Gamma(\nu/2)} K_{\nu/2}(\|(\nu\Lambda)^{1/2}t\|) \quad (3.1)$$

where $K_{\nu/2}(\|(\nu\Lambda)^{1/2}t\|)$ is the Macdonald function with order $\nu/2$ and argument $\|(\nu\Lambda)^{1/2}t\|$.

Proof. Set $Z = \Lambda^{-1/2}(X - \theta)$. Then the characteristic function of Z is given by

$$\phi_Z(t) = C(\nu, p) \int_{\mathbb{R}^p} \exp(it'z)(1 + z'z/\nu)^{(\nu+p)/2} dz.$$

An orthogonal transformation $Y = PZ$ with $y_1 = t'z(\|t\|)^{-1}$ gives

$$\phi_Z(t) = C(\nu, p) \int_{\mathbb{R}^p} \exp(i\|t\|y_1)(1 + y'y/\nu)^{(\nu+p)/2} dy. \quad (3.2)$$

Now, successive integration over $y_p, y_{p-1}, \dots, y_2$ in (3.2) yields

$$\begin{aligned} \phi_Z(t) &= C(\nu, 1) \int_{-\infty}^{\infty} (\cos\|t\|y_1 + i \sin\|t\|y_1)(1 + y_1^2/\nu)^{-(\nu+1)/2} dy_1 \\ &= 2C(\nu, 1) \int_0^{\infty} \cos(\|t\|y_1)(1 + y_1^2/\nu)^{-(\nu+1)/2} dy_1. \end{aligned}$$

By virtue of (2.1), we immediately have

$$\phi_Z(t) = \frac{\|\nu^{1/2}t\|^{\nu/2}}{2^{\nu/2-1}\Gamma(\nu/2)} K_{\nu/2}(\|\nu^{1/2}t\|).$$

Hence, the proof of the theorem is completed by noting that

$$\phi_X(t) = E \left[\exp \left(it'(\theta + \Lambda^{1/2}Z) \right) \right] = \exp(it'\theta) E[i(\Lambda^{1/2}t)'Z].$$

As a consequence of the series representation of Macdonald function given by (2.3) and (2.5), it is easily verified from (3.1) that the characteristic function of the multivariate t -distribution has the following series representations.

Case (i) ν positive and even

$$\begin{aligned} \phi_X(t) = & \exp(it'\theta) \left[\sum_{j=0}^{\nu/2-1} C_1(j) \|(\nu\Lambda)^{1/2}t\|^{2j} + \sum_{j=0}^{\infty} C_2(j) \|(\nu\Lambda)^{1/2}t\|^{\nu+2j} \right. \\ & \left. + \sum_{j=0}^{\infty} C_3(j) (-\ln(\|(\nu\Lambda)^{1/2}t\|)) \|(\nu\Lambda)^{1/2}t\|^{\nu+2j} \right] \end{aligned}$$

where

$$C_1(j) = \frac{(-1)^j (\nu/2 - 1 - j)!}{(\nu/2 - 1)! j! 4^j},$$

$$C_2(j) = \frac{\xi(1+j) + \xi(\nu/2 + 1 + j) + \ln 4}{2^\nu (\nu/2 - 1)! j! (\nu/2 + j)! 4^j},$$

$$C_3(j) = (2^{\nu-1} (\nu/2 - 1)! (\nu/2 + j)! 4^j)^{-1},$$

and $\xi(r)$ is defined by (2.4).

Case (ii) ν positive, and odd or fractional

$$\phi_X(t) = \exp(it'\theta) \sum_{j=0}^{\infty} [D_1(j) \|(\nu\Lambda)^{1/2}t\|^{2j} + D_2(j) \|(\nu\Lambda)^{1/2}t\|^{\nu+2j}]$$

where

$$D_1(j) = (j! 4^j (1 - \nu/2)_j)^{-1}$$

$$D_2(j) = \Gamma(-\nu/2) (2^\nu \Gamma(\nu/2) j! 4^j (1 + \nu/2)_j)^{-1}$$

with $(a)_j$, the Pochhammer polynomial, defined by (2.6).

4. CALCULATION OF MOMENTS

Let X have the multivariate t -distribution with p.d.f. given by (1.1). Then, it is well-known that $E(X) = \theta$ and $Cov(X) = \nu\Lambda/(\nu - 2)$. We now calculate product moment of the spherical t -distribution having p.d.f. (1.1) with $\Lambda = I$, an identity matrix.

Case (i) positive even ν

$$\begin{aligned} \phi_Y(t) = & \sum_{j=0}^{\nu/2-1} C_1(j) \nu^j (t't)^j + \sum_{j=0}^{\infty} C_2(j) \nu^{\nu/2+j} (t't)^{\nu/2+j} \\ & + C_3(j) (-\ln \sqrt{\nu} (t't)^{1/2}) \nu^{\nu/2+j} (t't)^{\nu/2+j} \end{aligned} \quad (4.1)$$

where $C_i(j)$'s ($i = 1, 2, 3$) are defined in Section 3.

Case (ii) positive noneven ν

$$\phi_Y(t) = \sum_{j=0}^{\infty} D_1(j) \nu^j (t't)^j + \sum_{j=0}^{\infty} D_2(j) \nu^{\nu/2+j} (t't)^{\nu/2+j} \quad (4.2)$$

where $D_i(j)$'s ($i = 1, 2$) are defined in Section 3.

To calculate $E(Y_1^2)$ for positive even ν , set $t_2 = t_3 = \dots = t_p = 0$ in (4.1) so that we have

$$\begin{aligned} \phi_Y(t) &= \sum_{j=0}^{\nu/2-1} C_1(j) \nu^j t_1^{2j} + \sum_{j=0}^{\infty} C_2(j) \nu^{\nu/2+j} t_1^{\nu+2j} \\ &\quad + C_3(j) (-\ln t_1 \sqrt{\nu}) \nu^{\nu/2+j} t_1^{\nu+2j}. \end{aligned}$$

After a tedious but straightforward algebra and by the use of the fact

$$\lim_{t_1 \downarrow 0} t_1^c \ln(t_1) = \lim_{t_1 \downarrow 0} \frac{\ln(t_1)}{t_1^{-c}} = \lim_{t_1 \downarrow 0} \frac{t_1^{-1}}{-ct_1^{-(c+1)}} = \lim_{t_1 \downarrow 0} \frac{t_1^c}{c} = 0, c > 0$$

we find that

$$i^2 E(Y_1^2) = \frac{d^2}{dt_1^2} \phi_Y(t_1, 0, \dots, 0) |_{t_1=0} = 2\nu C_1(1) = \nu/(\nu-2), \nu > 2.$$

The following results may be obtained from (4.1) for positive even values of ν by similar routine calculation.

$$\begin{aligned} E(Y_1^4) &= 24\nu^2 C_1(2) = \frac{3\nu^2}{(\nu-2)(\nu-4)}, \nu > 4, \\ E(Y_1^2 Y_2^2) &= 8\nu^2 C_1(2) = \frac{\nu^2}{(\nu-2)(\nu-4)}, \nu > 4 \text{ and} \\ E(Y_1^2 Y_2^2 Y_3^2) &= -48\nu^3 C_1(3) = \frac{\nu^3}{(\nu-2)(\nu-4)(\nu-6)}, \nu > 6. \end{aligned}$$

When ν is positive non-even we have, by differentiating (4.2), the same formulae for product moments as above except that C_1 is replaced by D_1 . In fact, it may be proved that for at least one odd m_k ($k = 1, 2, \dots, p$),

$$E\left(\prod_{k=1}^p Y_k^{m_k}\right) = 0 \text{ and for all even } m_k (k = 1, 2, \dots, p) \text{ with } m = \sum_{k=1}^p m_k,$$

the following result holds

$$E\left(\prod_{k=1}^p Y_k^{m_k}\right) = \begin{cases} \frac{(m/2)!(-4\nu)^{m/2} C_1(m/2)}{\pi^{p/2}} \left\{ \prod_{k=1}^p \left(\frac{m_k+1}{2} \right) \right\} & \text{for positive even } \nu(>m), \\ \frac{(m/2)!(-4\nu)^{m/2} D_1(m/2)}{\pi^{p/2}} \left\{ \prod_{k=1}^p \left(\frac{m_k+1}{2} \right) \right\} & \text{for positive non-even } \nu(>m), \end{cases}$$

where C_1 and D_1 are defined in Section 3. Finally, by setting the values of C_1 and D_1 , we obtain that the well-known result for the product moment of the Spherical t -Distribution is given by

$$E \left(\prod_{k=1}^p Y_k^{m_k} \right) = \begin{cases} 0 & \text{if at least one } m_k (k = 1, 2, \dots, p) \text{ is odd,} \\ \nu^{m/2} \frac{\Gamma((\nu-m)/2)}{2^m \Gamma(\nu/2)} \prod_{k=1}^p \frac{m_k!}{(m_k/2)!}, & \nu > m, \\ \text{if all } m_k s (k = 1, 2, \dots, p) \text{ are even} \end{cases} \quad (4.3)$$

(cf. Sutradhar, 1986).

5. A LIST OF CHARACTERISTIC FUNCTIONS

We now give a list of t -type distributions along with their characteristic functions expressed in terms of the Macdonald function. In what follows the normalizing constant $C(\nu, p)$ is given by (1.2).

(i) The Univariate Student t -distribution

$$f(x) = C(\nu, 1) \nu^{-1/2} (1 + x^2/\nu)^{-(\nu+1)/2}, x \in \mathbb{R}, \nu > 0,$$

$$\phi_X(t) = \frac{\nu^{1/4} |t|^{\nu/2}}{2^{\nu/2-1} \Gamma(\nu/2)} K_{\nu/2}(\nu^{1/2} |t|).$$

(ii) The Multivariate Cauchy Distribution

$$f(x) = C(1, p) |\Lambda|^{-1/2} (1 + (x - \theta)' \Lambda^{-1} (x - \theta))^{-(p+1)/2}, x \in \mathbb{R}^p,$$

$$\phi_X(t) = \exp(it' \theta) \frac{\|\Lambda^{1/2} t\|^{1/2}}{2^{1/2-1} \Gamma(1/2)} K_{1/2}(\|\Lambda^{1/2} t\|) = \exp(it' \theta - \|\Lambda^{1/2} t\|)$$

by virtue of (2.2) (cf. Press, 1982, p 175).

(iii) The Pearson Type VII Distribution (cf. Johnson and Kotz, 1970, p 114)

$$f(x) = C(2m - p, p) |\beta \Lambda|^{-1/2} (1 + (x - \theta)' (\beta \Lambda)^{-1} (x - \theta))^{-m},$$

$$x \in \mathbb{R}^p, m \geq p/2, \beta > 0,$$

$$\phi_X(t) = \exp(it' \theta) \frac{\|(\beta \Lambda)^{1/2} t\|^{m-p/2}}{2^{m-p/2-1} \Gamma(m-p/2)} K_{m-p/2}(\|(\beta \Lambda)^{1/2} t\|).$$

(iv) Multivariate t -type Distribution

$$f(x) = C(\nu, p) |\beta\Lambda|^{-1/2} (1 + (x - \theta)'(\beta\Lambda)^{-1}(x - \theta))^{-(\nu+p)/2}, x \in \mathbb{R}^p, \nu, \beta > 0,$$

$$\phi_X(t) = \exp(it'\theta) \frac{\|(\beta\Lambda)^{1/2}t\|^{\nu/2}}{2^{\nu/2-1}\Gamma(\nu/2)} K_{\nu/2}(\|(\beta\Lambda)^{1/2}t\|).$$

The parameter β is not necessarily a function of ν (cf. Rao, 1973, pp. 169-170). In particular, when $\beta = \nu - 2$ the above probability density function defines the mean-variance representation of the multivariate t -distribution (see Sutradhar, 1990).

It may be remarked that the well-known fact that the multivariate t -distribution can be written as a scale mixture of the normal distribution $N(\theta, \tau^2\Lambda)$ and the gamma distribution with scale parameter $\nu/2$ and mean 1 (i.e. $G(\nu/2, 2/\nu)$) follows from an integral representation of the Macdonald function given by Lebedev (1965, p 119) and the symmetry property of the Macdonald function with respect to order. We also note that as $\nu \rightarrow \infty$, the mixing distribution $G(\nu/2, 2/\nu)$ converges to a degenerate distribution with all the nonzero mass at the point unity explaining that the well-known fact that the limiting distribution of the multivariate t -distribution is the multivariate normal distribution.

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