

A Bivariate Chisquare Distribution and Some of its Properties

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Abstract A correlated bivariate chisquare distribution is derived from the bivariate Wishart distribution. A theorem is established that provides elegant expression for infinite sum of the product of several gamma functions. Then it is utilized for deriving product moments of the distribution. In particular, moments of the product and ratio of two correlated chisquare variables are derived. Some applications of these moments are outlined.

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1. Introduction

Let $X_1, X_2, \dots, X_N$ ($N > 2$) be a two-dimensional independent normal random vector with mean vector $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ so that the sums of squares and cross product matrix is given by

$\sum_{j=1}^N (X_j - \bar{X})(X_j - \bar{X})' = A$. Let the matrix A for the bivariate case be denoted by

$A = (a_{ik}), i = 1, 2; k = 1, 2$ where $a_{ii} = ms_i^2 = \sum_{j=1}^N (X_{ij} - \bar{X}_i)^2, m = N - 1, (i = 1, 2)$ and

$a_{12} = \sum_{j=1}^N (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mrs_1s_2$. Also let the elements of the matrix $\Sigma = (\sigma_{ik}),$

$i = 1, 2; k = 1, 2$ where $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2, \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0, \sigma_2 > 0$. The quantity ρ ($-1 < \rho < 1$) is the product moment correlation coefficient between X_1 and X_2 .

Fisher (1915) derived the distribution of the bivariate Wishart matrix in order to study the distribution of correlation coefficient for a bivariate normal sample. Wishart (1928) obtained the distribution of Wishart matrix as the joint distribution of sample variances and covariances from the multivariate normal population. The bivariate matrix A is said to have a Wishart distribution with parameters $m = N - 1$ and $\Sigma(2 \times 2) > 0$, written as $A \sim W_2(m, \Sigma)$ if its probability density function is given by

$$f_*(A) = \frac{|A|^{(m-3)/2} \exp\left(-\frac{1}{2} \text{tr} \Sigma^{-1} A\right)}{2^m \sqrt{\pi} |\Sigma|^{m/2} \prod_{i=1}^2 \Gamma((m+1-i)/2)}, \quad A > 0, m > 2$$

(See e.g. Anderson, 2003, 252).

Because of its important role in multivariate statistical analysis, various authors have given different derivations. See the references in Gupta and Nagar (2000, 87-88) for a good update on the moments of Wishart distribution.

In this paper we introduce a bivariate chisquare distribution of $U = mS_1^2 / \sigma_1^2$ and $V = mS_2^2 / \sigma_2^2$ based on the the bivariate Wishart distribution. Since the two variables are correlated, it is difficult to obtain product moments of the distribution. A theorem is established that provides elegant expressions for infinite sums of the product of gamma functions. Then it is utilized for deriving the product moments of the distribution and some other applications. In particular, moments of the product and ratio of two correlated chisquare variables are also derived.

The details of the formidable algebraic manipulations are omitted. In case the correlation coefficient $\rho = 0$, the probability density function (pdf) of U and V becomes the product of two independent chisquare variables each with m degrees of freedom, and the product moments obtained for the correlated case are found to be in agreement with the special situation of independence. We refer to Kotz, Balakrishnan and Johnson (2000) for other type of bivariate chisquare distributions. For any nonnegative integer k , the following notations will be used in sequel:

$$a_{\{k\}} = a(a+1)(a+2)\cdots(a+k-1),$$

$$a^{\{k\}} = a(a-1)\cdots(a-k+1).$$

2. The Density Function of the Bivariate Chisquare Distribution

The first three of the theorems are due to Joarder (2005).

Theorem 2.1 The random variables U and V are said to have a correlated bivariate chisquare distribution each with m degrees of freedom, if its probability density function is given by

$$(i) f(u, v) = \frac{(uv)^{m/2-1} e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m \sqrt{\pi} \Gamma(m/2) (1-\rho^2)^{m/2}} \sum_{k=0}^{\infty} \left(\frac{\rho \sqrt{uv}}{1-\rho^2} \right)^k \frac{\Gamma((k+1)/2)}{k! \Gamma((k+m)/2)}, \quad (2.1)$$

$$(ii) f(u, v) = \frac{(1-\rho^2)^{-m/2} (uv)^{m/2-1}}{2^m \Gamma^2(m/2)} \exp\left(-\frac{u+v}{2(1-\rho^2)}\right) E\left[\exp\left(\frac{\rho \sqrt{uv} \sqrt{Y}}{1-\rho^2}\right)\right]$$

$m = N - 1 > 2$, $-1 < \rho < 1$ and where Y has a beta distribution $B(a, b)$ with parameters $a = 1/2$ and $b = (m-1)/2$.

Proof. The pdf of the elements of A can be written as

$$f_1(a_{11}, a_{22}, a_{12}) = \frac{(1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{2^m \sqrt{\pi} \Gamma(m/2) \Gamma((m-1)/2)} (a_{11} a_{22} - a_{12}^2)^{(m-3)/2} \times \exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2} - \frac{a_{22}}{2(1-\rho^2)\sigma_2^2} + \frac{\rho a_{12}}{2(1-\rho^2)\sigma_1 \sigma_2}\right) \quad (2.2)$$

$a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, m > 2, -1 < \rho < 1$ (Anderson, 2003, 123). Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1 s_2$ with Jacobian $m^3 s_1 s_2$, the joint pdf of S_1^2, S_2^2 and R is given by

$$f_2(s_1^2, s_2^2, r) = \frac{m^m (1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{2^m \sqrt{\pi} \Gamma(m/2) \Gamma((m-1)/2)} s_1^{m-2} \exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right) \times s_2^{m-2} \exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right) (1-r^2)^{(m-3)/2} \exp\left(-\frac{m\rho r s_1 s_2}{2(1-\rho^2)\sigma_1 \sigma_2}\right). \quad (2.3)$$

By making the transformation $ms_1^2 = \sigma_1^2 u, ms_2^2 = \sigma_2^2 v$, with Jacobian

$J(s_1^2, s_2^2 \rightarrow u, v) = (\sigma_1 \sigma_2 / m)^2$, we have

$$f_3(u, v, r) = \frac{(1-r^2)^{(m-3)/2} (uv)^{m/2-1}}{4\pi \Gamma(m-1) (1-\rho^2)^{m/2}} \exp\left[-\frac{1}{2} \left(\frac{u+v-2\rho r \sqrt{uv}}{1-\rho^2} \right)\right].$$

By the use of $y = m-1$ in the duplication formula of gamma function

$$\Gamma(y) = 2^{y-1} \Gamma(y/2) \Gamma((y+1)/2) / \sqrt{\pi}$$

the probability density function turns out to be

$$f_4(u, v) = \frac{(uv)^{m/2-1} e^{-\frac{1}{2}(u+v)/(1-\rho^2)}}{4\pi \Gamma(m-1) (1-\rho^2)^{m/2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho \sqrt{uv}}{1-\rho^2} \right)^k \int_0^1 y^{(k-1)/2} (1-y)^{(m-3)/2} dy$$

which can be evaluated as

$$f(u, v) = \frac{(uv)^{m/2-1} e^{-\frac{1}{2}(u+v)/(1-\rho^2)}}{4\pi \Gamma(m-1) (1-\rho^2)^{m/2}} \sum_{k=0}^{\infty} \left(\frac{\rho \sqrt{uv}}{1-\rho^2} \right)^k \frac{\Gamma((k+1)/2) \Gamma((m-1)/2)}{k! \Gamma((m+k)/2)}.$$

Then by the use of the duplication formula of gamma function we have part (i) of the theorem. Part (ii) of the theorem is obvious.

Theorem 2.2 For $m > 2 \max(a, b)$ and $-1 < \rho < 1$, the (a, b) -th product moment $E(U^a V^b)$ of the distribution of U and V , is given by

$$\mu'(a, b; \rho) = \frac{2^{a+b} (1-\rho^2)^{a+b+m/2}}{\sqrt{\pi} \Gamma(m/2)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2} + a\right) \Gamma\left(\frac{k+m}{2} + b\right) \frac{\Gamma((k+1)/2)}{\Gamma((k+m)/2)}.$$

Proof. It follows from (2.1) that the (a, b) th product moment of the distribution of U is given by

$$E(U^a V^b) = \frac{(1-\rho^2)^{-m/2}}{2^m \sqrt{\pi} \Gamma(m/2)} \times \sum_{k=0}^{\infty} \left(\frac{\rho}{1-\rho^2} \right)^k \frac{1}{k!} \frac{\Gamma((k+1)/2)}{\Gamma((k+m)/2)} \int_0^{\infty} \int_0^{\infty} u^{\frac{m+k}{2}+a-1} v^{\frac{m+k}{2}+b-1} e^{-\frac{(u_1+u_2)}{2(1-\rho^2)}} du dv.$$

The theorem then follows by having evaluated the above integrals.

Since the product moments in Theorem 2.2 can be written as

$$\mu'(a, b; \rho) = \frac{2^{a+b} (1-\rho^2)^{a+b}}{L(m, \rho)} \sum_{k=0}^{\infty} \left(\frac{k+m}{2} \right)_{\{a\}} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2} + b\right) \frac{\Gamma((k+1)/2)}{\Gamma((k+m)/2)},$$

we need the following theorem to facilitate the calculation of moments for specific cases.

Theorem 2.3 Let $b_{k,m} = \frac{2^k}{k!} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{k+m}{2}\right)$ and $L(m, \rho) = \sqrt{\pi} \Gamma(m/2) (1-\rho^2)^{-m/2}$.

Then for $m > 2, -1 < \rho < 1$ and, we have

$$\begin{aligned} (i) \sum_{k=0}^{\infty} \rho^k b_{k,m} &= L(m, \rho) \\ (ii) \sum_{k=0}^{\infty} k b_{k,m} &= m \rho^2 (1-\rho^2)^{-1} L(m, \rho) = w_{(1)}(m, \rho) L(m, \rho) \\ (iii) \sum_{k=0}^{\infty} k^{\{2\}} \rho^k b_{k,m} &= (m(m+1)\rho^4 + m\rho^2)(1-\rho^2)^{-2} L(m, \rho) = w_{(2)}(m, \rho) L(m, \rho) \\ (iv) \sum_{k=0}^{\infty} k^{\{3\}} \rho^k b_{k,m} &= w_{(3)}(m, \rho) L(m, \rho) \text{ where} \\ w_{(3)}(m, \rho) &= ((m^3 + 3m^2 + 2m)\rho^6 + (3m^2 + 6m)\rho^4)(1-\rho^2)^{-3} \\ &= (m(m+1)(m+2)\rho^6 + 3m(m+2)\rho^4)(1-\rho^2)^{-3} L(m, \rho) = w_{(3)}(m) L(m, \rho) \\ (v) \sum_{k=0}^{\infty} k^{\{4\}} \rho^k b_{k,m} &= w_{(4)}(m, \rho) L(m, \rho) \text{ where} \\ w_{(4)}(m, \rho) &= [(m^4 + 6m^3 + 11m^2 + 6m)\rho^8 + (6m^3 + 30m^2 + 36m)\rho^6 + (3m^2 + 6m)\rho^4](1-\rho^2)^{-4} \\ &= [m(m+1)(m+2)(m+3)\rho^8 + 6m(m+2)(m+3)\rho^6 + 3m(m+2)\rho^4](1-\rho^2)^{-4} \\ (vi) \sum_{k=0}^{\infty} k^2 \rho^k b_{k,m} &= (m^2 \rho^4 + 2m \rho^2)(1-\rho^2)^{-2} L(m, \rho) = w_{(2)}(m, \rho) L(m, \rho) \\ (vii) \sum_{k=0}^{\infty} k^3 \rho^k b_{k,m} &= (m^3 \rho^6 + (6m^2 + 4m)\rho^4 + 4m \rho^2)(1-\rho^2)^{-3} L(m, \rho) = w_{(3)}(m, \rho) L(m, \rho) \\ (viii) \sum_{k=0}^{\infty} k^4 \rho^k b_{k,m} &= w_{(4)}(m, \rho) L(m, \rho) \text{ where} \\ w_{(4)}(m, \rho) &= [m^4 \rho^8 + (12m^3 + 16m^2 + 8m)\rho^6 + (28m^2 + 32m)\rho^4 + 8m \rho^2](1-\rho^2)^{-4}. \end{aligned}$$

For any nonnegative integer a , $\mu'(a, 0; \rho) = \frac{2^a \Gamma(m/2 + a)}{\Gamma(m/2)} = 2^a (m/2)_{\{a\}}$ which is the a -th moment of usual chisquare distribution with m degrees of freedom. Similarly,

$\mu'(0, b; \rho) = 2^b (m/2)_{(b)}$. When both orders a and b are negative, it is difficult to get closed form expressions for product moments $\mu'(a, b; \rho) = E(U^a V^b)$. Some other special cases are evaluated below from Theorem 2.2 by applying Theorem 2.3.

$$\mu'(1, 1; \rho) = m(m + 2\rho^2)$$

$$\mu'(1, 2; \rho) = m(m + 2)(m + 4\rho^2)$$

$$\mu'(1, 3; \rho) = m(m + 2)(m + 4)(m + 6\rho^2)$$

$$\mu'(1, 4; \rho) = m(m + 2)(m + 4)(m + 6)(m + 8\rho^2)$$

$$\mu'(2, 2; \rho) = m(m + 2)[8\rho^4 + 8(m + 2)\rho^2 + m(m + 2)]$$

$$\mu'(2, 3; \rho) = m(m + 2)(m + 4)[24\rho^4 + 12(m + 2)\rho^2 + m(m + 2)]$$

$$\mu'(2, 4; \rho) = m(m + 2)(m + 4)(m + 6)[48\rho^4 + 16(m + 2)\rho^2 + m(m + 2)]$$

$$\mu'(3, 3; \rho) = m(m + 2)(m + 4)[48\rho^6 + 72(m + 4)\rho^4 + 18(m + 2)(m + 4)\rho^2 + m(m + 2)(m + 4)]$$

$$\mu'(3, 4; \rho) = m(m + 2)(m + 4)(m + 6)$$

$$\times [192\rho^6 + 144(m + 4)\rho^4 + 24(m + 2)(m + 4)\rho^2 + m(m + 2)(m + 4)]$$

$$\mu'(4, 4; \rho) = m(m + 2)(m + 4)(m + 6)[384\rho^8 + 768(m + 6)\rho^6 + 288(m + 4)(m + 6)\rho^4 + 32(m + 2)(m + 4)(m + 6)\rho^2 + m(m + 2)(m + 4)(m + 6)]$$

Notice that if $\rho = 0$, $\mu'(a, b; \rho) = \mu'(a, 0; \rho)\mu'(0, b; \rho)$. In particular, if $\rho = 0$, then

$\mu'(4, 4; \rho) = m^2(m + 2)^2(m + 4)^2(m + 6)^2$ which is the product moment of order (4,4) of bivariate independent chisquare distribution each with m degrees of freedom. It is observed that if b is a nonnegative integer, then $\mu'(1, b; \rho) = 2^b (m/2)_{(b)}(m + 2b\rho^2)$.

Important centered product moments of bivariate chisquare distribution are derived below with the help of the raw product moments.

$$\mu(1, 1) = 2m\rho^2,$$

$$\mu(2, 0) = 2m,$$

$$\mu(2, 2) = (8\rho^4 + 4)m^2 + (16\rho^4 + 32\rho^2)m,$$

$$\mu(3, 1) = 12\rho^2m^2 + 48\rho^2m,$$

$$\mu(3, 3) = 120\rho^4m^4 + (48\rho^6 + 72\rho^2)m^3 + (288\rho^6 + 1152\rho^4 + 576\rho^2 + 64)m^2 + (384\rho^6 + 2304\rho^4 + 1152\rho^2)m,$$

$$\mu(4, 0) = 12m^2 + 48m,$$

$$\mu(4, 2) = (96\rho^4 + 24)m^3 + (1344\rho^4 + 2176\rho^2 + 96)m^2 + (2304\rho^4 + 1536\rho^2)m.$$

$$\mu(5, 1) = 120\rho^2m^3 + 2080\rho^2m^2 + 3840\rho^2m,$$

$$\mu(6, 0) = 120m^3 + 2080m^2 + 3840m.$$

3. Some Other Moments

Corollary 3.1 Let $\mu'(a, b, l; \rho) = E(S_1^{2a} S_2^{2b} R^l)$ be the product moments of the bivariate Wishart distribution with pdf given by (2.2). Then the (a, b) -th product moment of sample variances S_1^2 and S_2^2 can be calculated by

$$E(S_1^{2a} S_2^{2b}) = \mu'(a, b, 0; \rho) = \frac{\sigma_1^{2a} \sigma_2^{2b}}{m^{a+b}} \mu'(a, b; \rho).$$

The following corollary is obvious from Theorem 2.2.

Corollary 3.2 Let U and V have a bivariate chisquare distribution with pdf given by (2.1). Also let $\mu(a, b) = E((U - EU)^a (V - EV)^b)$ be the centered product moment between U and V . If a and b are of the same sign, then

- (i) $E(U^a V^b) = E(U^b V^a)$
- (ii) $\mu(a, b) = \mu(b, a)$.

Corollary 3.3 For $m > 2, -1 < \rho < 1$, the a th moment of $W = UV$ is given by

$$E(W^a) = \frac{4^a (1 - \rho^2)^{2a}}{L(m, \rho)} \sum_{k=0}^{\infty} \binom{k+m}{2}_{\{a\}} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2} + a\right) \frac{\Gamma((k+1)/2)}{\Gamma((k+m)/2)}$$

where $L(m, \rho)$ is defined in Theorem 2.3 (cf. Joarder, 2005a).

For $m > 2, -1 < \rho < 1$, the first four raw moments of W are given by

- (i) $E(W) = m(m + 2\rho^2)$,
- (ii) $E(W^2) = m(m + 2)[8\rho^4 + 8(m + 2)\rho^2 + m(m + 2)]$,
- (iii) $E(W^3) = 16(m/2)_{\{3\}}[24\rho^6 + 36(m + 4)\rho^4 + 9(m + 2)(m + 4)\rho^2 + 4(m/2)_3]$,
- (iv) $E(W^4) = 256\left(\frac{m}{2}\right)_{\{4\}} \left[24\rho^8 + 48(m + 6)\rho^6 + 72\left(\frac{m + 4}{2}\right)_{\{2\}} \rho^4 + 16\left(\frac{m + 2}{2}\right)_{\{3\}} \rho^2 + \left(\frac{m}{2}\right)_{\{4\}}\right]$.

The variance of W is given by

$$\text{Var}(W) = m[4(1 + \rho^2)m^2 + 4(1 + 8\rho^2 - \rho^4)m + 16\rho^4].$$

In case $\rho = 0$, then W will be the product of two independent chisquare random variables each with m degrees of freedom and evidently the resulting moments are in agreement with that situation. In particular, if $\rho = 0$, the mean and variance of W will be $E(W) = m$ and $\text{Var}(W) = 4m^2(m + 1)$ respectively.

Since $S_1^2 S_2^2 = \frac{\sigma_1^2 \sigma_2^2}{m^2} W$, the following corollary is obvious from Corollary 2.3.

Corollary 3.4 The mean and variance of $S_1^2 S_2^2$ are given by

$$E(S_1^2 S_2^2) = \frac{1}{m} (m + 2\rho^2) \sigma_1^2 \sigma_2^2 \text{ and}$$

$$Var(S_1^2 S_2^2) = \frac{1}{m^3} [4(1 + \rho^2)m^2 + 4(1 + 8\rho^2 - \rho^4)m + 16\rho^4] \sigma_1^4 \sigma_2^4$$

respectively.

Since the geometric mean of the chisquare variables are important in many applications, we define $G = (UV)^{1/2}$, the geometric mean of U and V , and provide its moments below.

Corollary 3.5 For $m > 2, -1 < \rho < 1$, the first four raw moments of G is given by

$$(i) E(G) = \frac{2(1 - \rho^2)^{m/2+1}}{\sqrt{\pi} \Gamma(m/2)} \sum_{k=0}^{\infty} \rho^k b_{k,m+1} \frac{\Gamma((k+m+1)/2)}{\Gamma((k+m)/2)},$$

$$(ii) E(G^2) = m(m + 2\rho^2),$$

$$(iii) E(G^3) = \frac{8(1 - \rho^2)^{m/2+3}}{\sqrt{\pi} \Gamma(m/2)} \sum_{k=0}^{\infty} \rho^k b_{k,m+3} \frac{\Gamma((k+m+3)/2)}{\Gamma((k+m)/2)},$$

$$(iv) E(G^4) = m(m+2)[8\rho^4 + 8(m+2)\rho^2 + m(m+2)].$$

Let $H = U/V$, the ratio of two correlated chisquare variables U and V that have probability density function in Theorem 2.1. Then the following corollary follows from Theorem 2.2.

Corollary 3.6 For $m > 2a, -1 < \rho < 1$, the a -th moment of H is given by

$$E(H^a) = \frac{1}{L(m, \rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2} + a\right) \Gamma\left(\frac{k+m}{2} - a\right) \Gamma\left(\frac{k+1}{2}\right) \left(\Gamma\left(\frac{k+m}{2}\right)\right)^{-1}$$

where $L(m, \rho)$ is defined in Theorem 2.3.

For $m > 8, -1 < \rho < 1$, the first four moments of H are given below:

$$(i) E(H) = \frac{m - 2\rho^2}{m - 2}, \quad m > 2,$$

$$(ii) E(H^2) = \frac{1}{(m-2)(m-4)} (24\rho^4 - 8(m+2)\rho^2 + m(m+2)), \quad m > 4,$$

$$(iii) E(H^3) = (2^3(m/2)^{\{3\}})^{-1}$$

$$\times [-480\rho^6 + 144(m+4)\rho^4 - 18(m+2)(m+4)\rho^2 + m(m+2)(m+4)], \quad m > 6,$$

$$(iv) E(H^4) = \left(2^4(m/2)^{(4)}\right)^{-1} \left[13440\rho^8 - 3840(m+6)\rho^6 + 480(m+4)(m+6)\rho^4 - 256(m/2+1)_{\{3\}}\rho^2 + 8(m/2)_{\{3\}}\right], m > 8.$$

The variance of H is given by

$$Var(H) = \frac{4}{(m-2)^2(m-4)} [(1-\rho^2)m^2 - (1+4\rho^2-5\rho^4)m + 8\rho^2(1-\rho^2)], m > 4.$$

In case $\rho = 0$, then H will be the ratio of two independent chisquare variables each m degrees of freedom, and the a -th moment will be simply

$$E(H^a) = (m/2)_{\{a\}} / (m/2-a)_{\{a\}}, m > 2a$$

which are evidently in agreement with the situation. In particular, if $\rho = 0$, the mean and variance of H will be

$$E(H) = \frac{m}{m-2}, m > 2, \text{ and}$$

$$Var(H) = \frac{4m(m-1)}{(m-2)^2(m-4)}, m > 4$$

respectively. Since $\frac{S_1^2}{S_2^2} = \frac{\sigma_1^2}{\sigma_2^2} H$, the following corollary is obvious from Corollary 3.6.

Corollary 3.7 The mean and variance of S_1^2/S_2^2 are given by

$$E(S_1^2/S_2^2) = \frac{m-2\rho^2}{m-2} \sigma_1^{-2} \sigma_2^2 \text{ and}$$

$$Var(S_1^2/S_2^2) = \frac{4}{(m-2)^2(m-4)} [(1-\rho^2)m^2 - (1+4\rho^2-5\rho^4)m + 8\rho^2(1-\rho^2)] \sigma_1^4 \sigma_2^{-4}, m > 4.$$

respectively.

4. Mahalanobis Moments

Since $\mu(2,0) = 2m$, $\mu(1,1) = 2m\rho^2$, $\mu(0,2) = 2m$, the covariance matrix between U and V is given by

$$\Omega = \begin{pmatrix} 2m & 2m\rho^2 \\ 2m\rho^2 & 2m \end{pmatrix}.$$

The standardized moments or Mahalanobis moments of a bivariate chisquare distribution are given by $\beta_i = E(Q^i)$, ($i = 1, 2, 3, \dots$) where $Q = W' \Omega^{-1} W$, $W' = (U - m \quad V - m)$ is the Mahalanobis distance. Note that

$$\beta_1 = E(tr W' \Omega^{-1} W) = E(tr \Omega^{-1} W W') = tr[\Omega^{-1} E(W W')] = tr(\Omega^{-1} \Omega) = tr(I_2) = 2.$$

Obviously the first standardized moment for any distribution is the dimension of the random variable in question.

The coefficient of kurtosis and skewness (Kotz, Balakrishnan and Johnson, 2000, 77) for the bivariate chisquare distribution is given by $\beta_2 = E(W' \Omega^{-1} W)^2$ and $\beta_3 = E(W' \Omega^{-1} W)^3$ respectively.

Theorem 4.1 Let X and Y have a bivariate distribution with $E(X^a Y^b) = E(X^b Y^a)$ and the correlation between them is ρ , then

$$(i) \mu^2(0, 2)(1 - \rho^2)^2 E(Q^2) = 2\mu(4, 0) + (4\rho^2 + 2)\mu(2, 2) - 8\rho\mu(3, 1)$$

$$(ii) \mu^3(0, 2)(1 - \rho^2)^3 E(Q^3) = 2\mu(6, 0) - (8\rho^3 + 12\rho)\mu(3, 3) - 12\rho\mu(5, 1) + (24\rho^2 + 6)\mu(4, 2).$$

If X and Y are two random variables, then it is well known that the independence between X and Y implies uncorrelation, but the converse is sometimes true, e.g. in case X and Y have a bivariate normal distribution. The following corollary is another example along the line.

Corollary 4.1 Let U and V have a bivariate chisquare distribution with pdf given by (2.1). If the product moment correlation between U and V vanishes, then they are independent.

Proof. Since

$$E(UV) = \mu'(1, 1) = m(m + 2\rho^2), \quad E(U) = \mu'(1, 0) = m, \quad E(V) = \mu'(0, 1) = m,$$

$$E(U^2) = \mu'(2, 0) = m(m + 2), \quad E(V^2) = \mu'(0, 2) = m(m + 2),$$

it can be checked that

$$Var(U) = m(m + 2) - m^2 = 2m,$$

$$Var(V) = m(m + 2) - m^2 = 2m,$$

$$Cov(U, V) = E(UV) - E(U)E(V) = m(m + 2\rho^2) - m(m) = 2m\rho^2$$

and hence the product moment correlation between U and V is given by

$$Corr(U, V) = \frac{Cov(UV)}{[Var(U)Var(V)]^{1/2}} = \rho^2.$$

Note that if $\rho = 0$, the pdf in (2.1) becomes the product of that of the two independent chisquare distributions each with m degrees of freedom. In view of the above corollary we have the following:

Corollary 4.2 Let X and Y have a bivariate normal distribution with correlation coefficient ρ . If $h_1(X_1, X_2, \dots, X_N)$ and $h_2(Y_1, Y_2, \dots, Y_N)$ are two functions of $(X_1, X_2, \dots, X_N)$ and $(Y_1, Y_2, \dots, Y_N)$ respectively, with correlation coefficient $h(\rho)$ such that $h(\rho) = 0$ implies $\rho = 0$, then X and Y are independent.

Theorem 4.2 For $m > 0$, the standardized moments of the bivariate chisquare distribution with pdf in Theorem 2.1 is given by

$$\begin{aligned} (i) \quad m(1-\rho^4)^2 E(Q^2) &= 8m(1-\rho^4)^2 + 8(3+2\rho^2-11\rho^4+4\rho^6+2\rho^8), \\ (ii) \quad m^2(1+\rho^2)^3 E(Q^3) &= 48(1+\rho^2)^3 m^2 + 16(18\rho^6+126\rho^4+135\rho^2+37)m \\ &\quad + 192(5+16\rho^2+18\rho^4+2\rho^6). \end{aligned}$$

Proof. Since the correlation between U and V is $\rho_{U,V} = \rho^2$, we replace $\rho_{X,Y} = \rho$ by $\rho_{U,V} = \rho^2$ in the expressions in Theorem 3.1. Moreover since $E(U^a V^b) = E(U^b V^a)$, it follows from Theorem 4.1 that

$$\mu^2(0,2)(1-\rho^4)^2 E(Q^2) = 2 + \mu(4,0) + (4\rho^4+2)\mu(2,2) - 8\rho^2\mu(3,1), \quad (4.1)$$

$$\mu^3(0,2)(1-\rho^4)^3 E(Q^3) = 2\mu(6,0) - (8\rho^6+12\rho^2)\mu(3,3) - 12\rho^2\mu(5,1) + (24\rho^4+6)\mu(4,2). \quad (4.2)$$

Then part (i) follows from (4.1) in the following way:

$$\begin{aligned} &4m^2(1-\rho^4)^2 E(Q^2) \\ &= 2(12m^2+48m) + (4\rho^4+2)(8\rho^4m^2+16\rho^4m+32\rho^2m+4m^2) - 8\rho^2(12m^2\rho^2+48m\rho^2) \\ &= 4(8\rho^8-16\rho^4+8)m^2 + 4(16\rho^8+32\rho^6-88\rho^4+16\rho^2+24)m. \end{aligned}$$

From (4.2) we have

$$\begin{aligned} &(2m)^3 \left((1-\rho^2)(1+\rho^2) \right)^3 E(Q^3) \\ &= 2(120m^3+2080m^2+3840m) \\ &\quad - (8\rho^6+12\rho^2) \left[(48\rho^6+72\rho^2)m^3 + (288\rho^6+1152\rho^4+576\rho^2+64)m^2 + (384\rho^6+2304\rho^4+1152\rho^2)m \right] \\ &\quad - 12\rho^2(120\rho^2m^3+2080\rho^2m^2+3840\rho^2m) \\ &\quad + (24\rho^4+6)(96\rho^4m^3+1344\rho^4m^2+2304\rho^4m+2176\rho^2m^2+1536\rho^2m+24m^3+96m^2), \end{aligned}$$

which can be simplified as

$$\begin{aligned} &(2m)^3 \left((1-\rho^2)(1+\rho^2) \right)^3 E(Q^3) \\ &= (-384\rho^{12}+1152\rho^8-1152\rho^4+384)m^3 \\ &\quad + (-2304\rho^{12}-9216\rho^{10}+24192\rho^8+1024\rho^6-21504\rho^4+3072\rho^2+4736)m^2 \\ &\quad + (-3072\rho^{12}-18432\rho^{10}+41472\rho^8+9216\rho^6-46080\rho^4+9216\rho^2+7680)m, \end{aligned}$$

or,

$$\begin{aligned}
& m^2 \left((1-\rho^2)(1+\rho^2) \right)^3 E(Q^3) \\
& = (-48\rho^{12} + 144\rho^8 - 144\rho^4 + 48)m^3 \\
& + (-288\rho^{12} - 1152\rho^{10} + 3024\rho^8 + 128\rho^6 - 2688\rho^4 + 384\rho^2 + 592)m^2 \\
& + (-384\rho^{12} - 2304\rho^{10} + 5184\rho^8 + 1152\rho^6 - 5760\rho^4 + 1152\rho^2 + 960)m.
\end{aligned}$$

Dividing both sides by $(1-\rho^2)^3$, we have (ii).

In case $\rho = 0$, the pdf of the joint probability distribution in Theorem 2.1, would be product of two independent chisquare random variables given by

$$f(u, v) = \frac{(uv)^{m/2-1} e^{-(u+v)/2}}{2^m \Gamma^2(m/2)}, \quad u > 0, v > 0.$$

Corollary 4.3 For independent bivariate chisquare distribution with pdf given above we have

$$\begin{aligned}
(i) E(Q^2) &= 8 + 24/m, \\
(ii) E(Q^3) &= 48 + (592/m) + (960/m^2).
\end{aligned}$$

These moments coincide, as expected, to that of the bivariate normal distribution as m tends to infinity.

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