

Family of estimators of mean, ratio and product of a finite population using random nonresponse

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Received: 12 April 2004 / Accepted: 18 July 2005 /
Published online: 27 February 2007
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Abstract In this paper a family of estimators for estimating mean, ratio and product of two means of a finite population are suggested and studied under the two different situations of random non-response considered by Tracy and Osahan (1994, *Statistica* 54(2):163–168), Singh and Joarder (1998, *Metrika* 47:241–249) and Singh et al. (2000, *Statistica* 60(1):39–44). Asymptotic expressions of biases and mean squared errors of the proposed families are derived. Optimum conditions are obtained under which the proposed families of estimators have the minimum mean squared error (MSE). Furthermore, the optimum values, depending upon population parameters, when replaced by sample values, yield the estimators having the minimum MSE of the optimum estimators. The estimators for MSEs of the suggested families are also given.

Keywords Auxiliary information · Random non-response · Bias · Mean squared error

Mathematics Subject Classification (2000) 62D05

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1 Introduction

In sample surveys it is common to make use of auxiliary information to increase the precision of the estimates of population parameters. The problem of estimating the ratio and product of two means of a finite population using information on a single auxiliary variable or many of them has been discussed, among others, by Singh (1965, 1967, 1986a, 1986b, 1988), Rao and Pereira (1968), Shah and Shah (1978), Tripathi (1980), Ray and Singh (1985), Upadhyaya and Singh (1985), Upadhyaya et al. (1985), Srivastava et al. (1989) and Singh et al. (1994a, 1994b). Let y_i ($i = 0, 1$) be the study characters with population means $\bar{Y}_i$ ($i = 0, 1$), and let y_2 be an auxiliary character (correlated with study characters y_i ($i = 0, 1$)) with a known population mean $\bar{Y}_2$. Assume that a simple random sample of size n is drawn without replacement and (y_{0i}, y_{1i}, y_{2i}) , $i = 1, 2, \dots, n$, are observed. The usual estimator of the ratio $R_{(\alpha)} = \bar{Y}_0/\bar{Y}_1^\alpha$ ($\bar{Y}_1 \neq 0$) is defined by

$$\hat{R}_{(\alpha)} = \frac{\bar{y}_0}{\bar{y}_1^\alpha} \quad (\bar{y}_1 \neq 0), \quad (1.1)$$

where α is a scalar which takes values 0, 1 and -1 . It is to be mentioned that:

- (i) for $\alpha = 0$, $R_{(\alpha)} \rightarrow R_{(0)} = \bar{Y}_0$, and its estimator $\hat{R}_{(\alpha)} \rightarrow \hat{R}_{(0)} = \bar{y}_0$,
- (ii) for $\alpha = 1$, $R_{(\alpha)} \rightarrow R_{(1)} = \bar{Y}_0/\bar{Y}_1$, and its estimator $\hat{R}_{(\alpha)} \rightarrow \hat{R}_{(1)} = \bar{y}_0/\bar{y}_1 = \hat{R}$,
- (iii) for $\alpha = -1$, $R_{(\alpha)} \rightarrow R_{(-1)} = \bar{Y}_0\bar{Y}_1$, and its estimator $\hat{R}_{(\alpha)} \rightarrow \hat{R}_{(-1)} = \bar{y}_0\bar{y}_1 = \hat{P}$.

Utilizing the information on an auxiliary character y_2 , a class of estimators for $R_{(\alpha)}$ on the basis of Srivastava (1971) is defined by

$$\hat{R}_{(\alpha)}^{(h)} = \hat{R}_{(\alpha)} h(u), \quad (1.2)$$

where $h(u)$ is a function of $u = \bar{y}_2/\bar{Y}_2$ such that $h(1) = 1$ and satisfying certain regularity conditions, as mentioned in Srivastava (1971). Motivated by Srivastava (1980), we propose a wider class of estimators for $R_{(\alpha)}$ as

$$\hat{R}_{(\alpha)}^{(H)} = H(\hat{R}_{(\alpha)}, u), \quad (1.3)$$

where $H(\hat{R}_{(\alpha)}, u)$ is a function of $(\hat{R}_{(\alpha)}, u)$ such that $H(R_{(\alpha)}, 1) = R_{(\alpha)}$, $H_1(R_{(\alpha)}, 1) = 1$ and satisfying certain conditions, as given in Srivastava (1980).

Assuming the population mean $\bar{Y}_2$ and the variance S_2^2 of the auxiliary character y_2 to be known and following the same approach as adopted by Srivastava and Jhaji (1981) and Upadhyaya and Singh (1985), a family of estimators of $R_{(\alpha)}$ is defined by

$$\hat{R}_{(\alpha)}^{(t)} = \hat{R}_{(\alpha)} t(u, v), \quad (1.4)$$

where $t(u, v)$ is a function of (u, v) such that $t(1, 1) = 1$ and satisfying certain regularity conditions, as defined in the Appendix. For $\alpha = 1$, $\hat{R}_{(\alpha)}^{(t)}$ reduces to Upadhyaya

and Singh (1985) estimator of the ratio $R_{(1)} = R$ given by

$$\hat{R}_{(1)}^{(t)} = \hat{R} t(u, v), \quad (1.5)$$

while for $\alpha = 0$ it reduces to the Srivastava and Jhaji (1981) estimator of the population mean $\bar{Y}_0$

$$\hat{R}_{(0)}^{(t)} = \bar{y}_0 t(u, v). \quad (1.6)$$

To the first degree of approximation the biases and MSEs of $\hat{R}_{(\alpha)}^{(h)}$ and $\hat{R}_{(\alpha)}^{(t)}$ are, respectively, given by

$$B(\hat{R}_{(\alpha)}^{(h)}) = B(\hat{R}_{(\alpha)}) + \frac{\theta R_{(\alpha)}}{2} C_2^2 \{2K_{(\alpha)} h_1(1) + h_{11}(1)\}, \quad (1.7)$$

$$B(\hat{R}_{(\alpha)}^{(t)}) = B(\hat{R}_{(\alpha)}) + \frac{\theta R_{(\alpha)}}{2} [2K_{(\alpha)} C_2^2 t_1(1, 1) + 2d_{(\alpha)} t_2(1, 1) + C_2^2 t_{11}(1, 1) + 2\lambda_{003} C_2 t_{12}(1, 1) + (\lambda_{004} - 1) t_{22}(1, 1)], \quad (1.8)$$

$$\text{MSE}(\hat{R}_{(\alpha)}^{(h)}) = \text{MSE}(\hat{R}_{(\alpha)}) + \theta R_{(\alpha)}^2 C_2^2 h_1(1) \{h_1(1) + 2K_{(\alpha)}\}, \quad (1.9)$$

and

$$\begin{aligned} \text{MSE}(\hat{R}_{(\alpha)}^{(t)}) &= \text{MSE}(\hat{R}_{(\alpha)}) + \theta R_{(\alpha)}^2 [C_2^2 t_1^2(1, 1) + (\lambda_{004} - 1) t_2^2(1, 1) \\ &\quad + 2\lambda_{003} C_2 t_1(1, 1) t_2(1, 1) + 2K_{(\alpha)} C_2^2 t_1(1, 1) + 2d_{(\alpha)} t_2(1, 1)], \end{aligned} \quad (1.10)$$

where the bias and MSE of $\hat{R}_{(\alpha)}$, to the first degree of approximation, are, respectively, given by

$$B(\hat{R}_{(\alpha)}) = \theta C_1^2 R_{(\alpha)} (\alpha/2) [\alpha - 2K_{01} + 1] \text{ and } \text{MSE}(\hat{R}_{(\alpha)}) = \theta R_{(\alpha)}^2 A_{(\alpha)}. \quad (1.11)$$

The $\text{MSE}(\hat{R}_{(\alpha)}^{(h)})$ and $\text{MSE}(\hat{R}_{(\alpha)}^{(t)})$ in (1.9) and (1.10) are, respectively, minimized for

$$h_1(1) = -K_{(\alpha)} \quad \text{and} \quad t_1(1, 1) = A, \quad t_2(1, 1) = B. \quad (1.12)$$

Putting (1.12) in (1.9) and (1.10) yields the min.MSEs of $\hat{R}_{(\alpha)}^{(h)}$ and $\hat{R}_{(\alpha)}^{(t)}$ as

$$\text{min.MSE}(\hat{R}_{(\alpha)}^{(h)}) = \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 K_{(\alpha)}^2 C_2^2 \quad (1.13)$$

and

$$\text{min.MSE}(\hat{R}_{(\alpha)}^{(t)}) = \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 B_{(\alpha)}. \quad (1.14)$$

Thus, we have established the following theorems:

Theorem 1.1 Up to terms of order n^{-1} ,

$$\text{MSE}(\hat{R}_{(\alpha)}^{(h)}) \geq \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 K_{(\alpha)}^2 C_2^2,$$

with the equality holding if $h_1(1) = -K_{(\alpha)}$.

Theorem 1.2 Up to terms of order n^{-1} ,

$$\text{MSE}(\hat{R}_{(\alpha)}^{(t)}) \geq \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 B_{(\alpha)},$$

with the equality holding if $t_1(1, 1) = A$ and $t_2(1, 1) = B$.

Any parametric function $t(u, v)$, satisfying the regularity conditions, can generate an asymptotically acceptable estimator. The following estimators

$$\begin{aligned} d_{0(1)} &= \hat{R}_{(\alpha)} u^{\alpha_1} v^{\alpha_2}, \\ d_{0(2)} &= \hat{R}_{(\alpha)} \{1 + \alpha_1(u - 1)\} / \{1 + \alpha_2(v - 1)\}, \\ d_{0(3)} &= \hat{R}_{(\alpha)} [1 - \alpha_1(u - 1) - \alpha_2(v - 1)]^{-1}, \\ d_{0(4)} &= \hat{R}_{(\alpha)} [1 - \alpha_1(u - 1) - \alpha_2(v - 1)], \\ d_{0(5)} &= \hat{R}_{(\alpha)} (2 - u^{\alpha_1} v^{\alpha_2}), \quad \text{and} \\ d_{0(6)} &= \hat{R}_{(\alpha)} [\alpha_1 u + (1 - \alpha_1) v^{\alpha_2}], \quad \text{etc.} \end{aligned}$$

of the parameter $R_{(\alpha)}$ are members of the family of estimators $\hat{R}_{(\alpha)}^{(t)}$, where α_1 and α_2 are real constants. The optimum values of the constants α_1 and α_2 are obtained by the right hand sides of (1.12), and the resulting estimators will have the same minimum MSE given by (1.14). Furthermore, we note that the class of estimators $\hat{R}_{(\alpha)}^{(t)}$ does not include the simple difference-type estimator

$$d_{0(7)} = \hat{R}_{(\alpha)} + \alpha_1(u - 1) + \alpha_2(v - 1). \quad (1.15)$$

This led the authors to propose a class of estimators wider than $\hat{R}_{(\alpha)}^{(t)}$ in (1.4), as

$$\hat{R}_{(\alpha)}^{(T)} = T(\hat{R}_{(\alpha)}, u, v), \quad (1.16)$$

where $T(\hat{R}_{(\alpha)}, u, v)$ is a function of $(\hat{R}_{(\alpha)}, u, v)$ such that $T(D) = R_{(\alpha)}$, $T_1(D) = 1$ and satisfying certain regularity conditions, as defined in the Appendix. To the first degree of approximation the biases and MSEs of $\hat{R}_{(\alpha)}^{(H)}$ and $\hat{R}_{(\alpha)}^{(T)}$ are, respectively, given by

$$\begin{aligned} B(\hat{R}_{(\alpha)}^{(H)}) &= B(\hat{R}_{(\alpha)}) + \frac{\theta}{2} C_2^2 [H_{22}(R_{(\alpha)} - 1) \\ &\quad + 2R_{(\alpha)} K_{(\alpha)} H_{12}(R_{(\alpha)}, 1)], \end{aligned} \quad (1.17)$$

$$\begin{aligned} B(\hat{R}_{(\alpha)}^{(T)}) &= B(\hat{R}_{(\alpha)}) + \frac{\theta}{2} [C_2^2 T_{22}(D) + (\lambda_{004} - 1) T_{33}(D) \\ &\quad + 2R_{(\alpha)} K_{(\alpha)} T_{12}(D) C_2^2 + 2R_{(\alpha)} d_{(\alpha)} T_{13}(D) \\ &\quad + 2\lambda_{003} C_2 T_{23}(D)], \end{aligned} \quad (1.18)$$

$$\begin{aligned} \text{MSE}(\hat{R}_{(\alpha)}^{(H)}) &= \text{MSE}(\hat{R}_{(\alpha)}) + \theta C_2^2 H_2(R_{(\alpha)}, 1, 1) [H_2(R_{(\alpha)} 1, 1) \\ &\quad + 2R_{(\alpha)} K_{(\alpha)}], \end{aligned} \quad (1.19)$$

and

$$\begin{aligned} \text{MSE}(\hat{R}_{(\alpha)}^{(T)}) &= \text{MSE}(\hat{R}_{(\alpha)}) + \theta [C_2^2 T_2^2(D) + (\lambda_{004} - 1) T_3^2(D) \\ &\quad + 2R_{(\alpha)} K_{(\alpha)} C_2^2 T_2(D) + 2R_{(\alpha)} d_{(\alpha)} T_3(D) \\ &\quad + 2\lambda_{003} C_2 T_2(D) T_3(D)]. \end{aligned} \quad (1.20)$$

The $\text{MSE}(\hat{R}_{(\alpha)}^{(H)})$ and $\text{MSE}(\hat{R}_{(\alpha)}^{(T)})$ in (1.19) and (1.20), respectively, are minimized for

$$H_2(R_{(\alpha)}, 1) = -R_{(\alpha)} K_{(\alpha)} \quad (1.21)$$

and

$$T_2(D) = R_{(\alpha)} A, \quad T_3(D) = R_{(\alpha)} B. \quad (1.22)$$

Putting (1.21) and (1.22) in (1.19) and (1.20), respectively, we get the minimum MSEs of $\hat{R}_{(\alpha)}^{(H)}$ and $\hat{R}_{(\alpha)}^{(T)}$ as

$$\min. \text{MSE}(\hat{R}_{(\alpha)}^{(H)}) = \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2 = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(h)}) \quad (1.23)$$

and

$$\min. \text{MSE}(\hat{R}_{(\alpha)}^{(T)}) = \text{MSE}(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 B_{(\alpha)} = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(t)}). \quad (1.24)$$

Thus, it is proved that the asymptotic MSE for an optimum estimator of the family generated by $\hat{R}_{(\alpha)}^{(T)}$ ($\hat{R}_{(\alpha)}^{(H)}$) is same as the asymptotic MSE of an optimum estimator of the family $\hat{R}_{(\alpha)}^{(t)}$ ($\hat{R}_{(\alpha)}^{(h)}$) and is not further reduced.

From (1.11), (1.23) and (1.24) we have

$$\text{MSE}(\hat{R}_{(\alpha)}) - [\min. \text{MSE}(\hat{R}_{(\alpha)}^{(h)}) = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(H)})] = \theta R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2 \geq 0 \quad (1.25)$$

and

$$\begin{aligned} &[\min. \text{MSE}(\hat{R}_{(\alpha)}^{(h)}) = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(H)})] \\ &- [\min. \text{MSE}(\hat{R}_{(\alpha)}^{(t)}) = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(T)})] = \theta R_{(\alpha)}^2 \frac{\Delta_2}{\Delta} \geq 0. \end{aligned} \quad (1.26)$$

Thus, from (1.25) and (1.26) we state the following theorem:

Theorem 1.3 *The inequality*

$$\begin{aligned} &[\min. \text{MSE}(\hat{R}_{(\alpha)}^{(t)}) = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(T)})] \\ &\leq [\min. \text{MSE}(\hat{R}_{(\alpha)}^{(h)}) = \min. \text{MSE}(\hat{R}_{(\alpha)}^{(H)})] \leq \text{MSE}(\hat{R}_{(\alpha)}) \end{aligned}$$

holds under the optimum conditions.

Table 1 Values of the derivatives

Estimator	$t_1(1, 1)$	$t_2(1, 1)$	$t_{11}(1, 1)$	$t_{12}(1, 1)$	$t_{22}(1, 1)$
$d_{0(1)}$	α_1	α_2	$\alpha_1(\alpha_1 - 1)$	$\alpha_1\alpha_2$	$\alpha_2(\alpha_2 - 1)$
$d_{0(2)}$	α_1	$-\alpha_2$	0	$-\alpha_1\alpha_2$	$2\alpha_2^2$
$d_{0(3)}$	α_1	α_2	$2\alpha_1^2$	$2\alpha_1\alpha_2$	$2\alpha_2^2$
$d_{0(4)}$	$-\alpha_1$	$-\alpha_2$	0	0	0
$d_{0(5)}$	$-\alpha_1$	$-\alpha_2$	$-\alpha_1(\alpha_1 - 1)$	$-\alpha_1\alpha_2$	$-\alpha_2(\alpha_2 - 1)$
$d_{0(6)}$	α_1	$\alpha_2(1 - \alpha_1)$	0	0	$\alpha_2(1 - \alpha_1)(\alpha_2 - 1)$

It follows from Theorem 1.3 that the proposed family of estimators $\hat{R}_{(\alpha)}^{(t)}$ ($\hat{R}_{(\alpha)}^{(T)}$) is more efficient than the estimators $\hat{R}_{(\alpha)}$, $\hat{R}_{(\alpha)}^{(h)}$ and $\hat{R}_{(\alpha)}^{(H)}$. It is to be mentioned that $\hat{R}_{(\alpha)}^{(h)}$ is a member of the class $\hat{R}_{(\alpha)}^{(t)}$. The estimator $\hat{R}_{(\alpha)}^{(T)}$ includes all the estimators $\hat{R}_{(\alpha)}$, $\hat{R}_{(\alpha)}^{(h)}$, $\hat{R}_{(\alpha)}^{(H)}$ and $\hat{R}_{(\alpha)}^{(t)}$. The biases and MSEs of the estimators $d_{0(i)}$, $i = 1, \dots, 6$, can be obtained from (1.8) and (1.10) just by putting the values of the derivatives, as shown in Table 1.

In this paper we study the effect of a random non-response on the family of estimators $\hat{R}_{(\alpha)}^{(h)}$, $\hat{R}_{(\alpha)}^{(H)}$, $\hat{R}_{(\alpha)}^{(t)}$, and $\hat{R}_{(\alpha)}^{(T)}$ in the two situations advocated by Tracy and Osahan (1994): (i) a non-response in the study as well as the auxiliary variables [Situation 1], and (ii) a non-response in the study variable only [Situation 2]. The reader is referred to Singh and Joarder (1998), Singh and Singh (1979, 1985), Singh et al. (2000) and Singh and Tracy (2001).

2 A non-response probability model

Let Ω : $(v_1, v_2, \dots, v_N)$ be the population of N units from which a simple random sample of size n is drawn without replacement. If r ($r = 0, 1, 2, \dots, (n - 2)$) denotes the number of sampling units on which information could not be obtained due to random non-responses, then the remaining $(n - r)$ units in the sample can be treated as a simple random sampling without replacement (SRSWOR) from Ω . It is assumed that r is less than $(n - 1)$. We also assume that if p denotes the probability of a non-response among the $(n - 2)$ possible values of non-responses, then r has the following discrete distribution

$$P(r) = \frac{(n - r)}{nq + 2p} C_r^{n-2} p^r q^{n-r-2}, \quad (2.1)$$

which is due to Singh and Joarder (1998), where $q = 1 - p$ and $r = 0, 1, 2, \dots, (n - 2)$. Let us define

$$\begin{aligned} e_0 &= \frac{\bar{y}_{0(n-r)}}{\bar{Y}_0} - 1, & e_1 &= \frac{\bar{y}_{1(n-r)}}{\bar{Y}_1}, & e_2 &= \frac{\bar{y}_{2(n-r)}}{\bar{Y}_2} - 1, & e_3 &= \frac{\bar{y}_2}{\bar{Y}_2} - 1, \\ e_4 &= \frac{s_{2(n-r)}^2}{S_2^2} - 1, & e_5 &= \frac{s_2^2}{S_2^2} - 1. \end{aligned}$$

Incompleteness (in the form of absence) is a troublesome feature of many data sets. Statisticians have long known that failure to account for the stochastic nature of incompleteness can damage the actual conclusion. An obvious problem, that one needs to justify, arises when ignoring the incomplete mechanism. Rubin (1976) advocated three concepts: missing at random (MAR), observed at random (OAR), and parameter distribution (PD). Rubin defined: “The data are MAR if the probability of the observed missingness pattern, given the observed and unobserved data, does not depend on the value of the unobserved data”. Heitjan and Basu (1996) have distinguished the meaning of missing at random (MAR) and missing completely at random (MCAR) in a very nice manner. The probability model, defined in (2.1), is free from actual data values; hence, can be considered as a model suitable for MAR situation. Then, under the probability model given in (2.1), we have the following results:

$$\begin{aligned}
 E(e_i) &= 0, \quad i = 0, 1, \dots, 5; \\
 E(e_0^2) &= \theta^* C_0^2, \quad E(e_1^2) = \theta^* C_1^2, \quad E(e_2^2) = \theta^* C_2^2, \quad E(e_3^2) = \theta C_2^2, \\
 E(e_4^2) &= \theta^* (\lambda_{004} - 1), \quad E(e_5^2) = \theta (\lambda_{004} - 1), \\
 E(e_0 e_1) &= \theta^* \rho_{01} C_0 C_1, \quad E(e_0 e_2) = \theta^* \rho_{02} C_0 C_2, \quad E(e_0 e_3) = \theta \rho_{02} C_0 C_2, \\
 E(e_0 e_4) &= \theta^* \lambda_{102} C_0, \quad E(e_0 e_5) = \theta \lambda_{102} C_0, \\
 E(e_1 e_2) &= \theta^* \rho_{12} C_1 C_2, \quad E(e_1 e_3) = \theta \rho_{12} C_1 C_2, \\
 E(e_1 e_4) &= \theta^* \lambda_{012} C_1, \quad E(e_1 e_5) = \theta \lambda_{102} C_1, \\
 E(e_2 e_3) &= \theta C_2^2, \quad E(e_2 e_4) = \theta^* \lambda_{003} C_2, \quad E(e_2 e_5) = \theta \lambda_{003} C_2, \\
 E(e_3 e_4) &= \theta \lambda_{003} C_2, \quad E(e_3 e_5) = \theta \lambda_{003} C_2, \quad E(e_4 e_5) = \theta (\lambda_{004} - 1).
 \end{aligned}$$

It is to be noted that if $p = 0$, that is, there are no non-responses, the above expected values coincide with the usual results. For practical examples of random non-response in survey sampling, the reader is referred to Tracy and Osahan (1994) and Toutenburg and Srivastava (1998).

3 Proposed strategies

In this section we consider three different strategies depending upon the availability of information and non-responses on both variables.

3.1 Strategy I

When random non-responses for r units on study variables y_0, y_1 and auxiliary variable y_2 are present in the sample, and the population mean $\bar{Y}_2$ and the variance S_2^2 of y_2 are known, we define a family of estimators of $R_{(\alpha)}$ as

$$d_1 = \hat{R}_{\alpha(n-r)} f(u_{(n-r)}, v_{(n-r)}), \quad (3.1)$$

where $f(u_{(n-r)}, v_{(n-r)})$ is a function of $(u_{(n-r)}, v_{(n-r)})$ such that $f(1, 1) = 1$ and satisfying the following conditions:

- (i) Whatever the sample, $(u_{(n-r)}, v_{(n-r)})$ assumes values in a bounded, closed convex subset, $\mathbf{S}$, of the two-dimensional real space containing the point $(1, 1)$.
- (ii) In $\mathbf{S}$ the function $f(u_{(n-r)}, v_{(n-r)})$ is continuous and bounded.
- (iii) The first and second order partial derivatives of $f(u_{(n-r)}, v_{(n-r)})$ exist as well as are continuous and bounded in $\mathbf{S}$.

To the first degree of approximation the bias and MSE of d_1 are, respectively, given by

$$B(d_1) = B(\hat{R}_{\alpha(n-r)}) + \frac{\theta^* R_{(\alpha)}}{2} [2K_{(\alpha)} C_2^2 f_1(1, 1) + 2d_{(\alpha)} f_2(1, 1) + C_2^2 f_{11}(1, 1) + 2\lambda_{003} C_2 f_{12}(1, 1) + (\lambda_{004} - 1) f_{22}(1, 1)] \quad (3.2)$$

and

$$\text{MSE}(d_1) = \text{MSE}(\hat{R}_{\alpha(n-r)}) + \theta^* R_{(\alpha)}^2 [C_2^2 f_1^2(1, 1) + (\lambda_{004} - 1) f_2^2(1, 1) + 2\lambda_{003} C_2 f_1(1, 1) f_2(1, 1) + 2K_{(\alpha)} C_2^2 f_1(1, 1) + 2d_{(\alpha)} f_2(1, 1)], \quad (3.3)$$

where

$$B(\hat{R}_{\alpha(n-r)}) = \theta^* R_{(\alpha)} \frac{\alpha C_1^2}{2} (\alpha - 2K_{01} + 1) \quad (3.4)$$

and

$$\text{MSE}(\hat{R}_{\alpha(n-r)}) = \theta^* R_{(\alpha)}^2 [C_0^2 + \alpha C_1^2 (\alpha - 2K_{01})] \quad (3.5)$$

are, respectively, the bias and MSE of $\hat{R}_{\alpha(n-r)}$ to the first degree of approximation. The optimum values of $f_1(1, 1)$ and $f_2(1, 1)$, for which $\text{MSE}(d_1)$ is minimum, are given by

$$f_1(1, 1) = A \quad \text{and} \quad f_2(1, 1) = B. \quad (3.6)$$

Substitution of (3.6) into (3.3) yields the minimum MSE of d_1 as

$$\text{min.MSE}(d_1) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 B_{(\alpha)}. \quad (3.7)$$

Thus, we state the following theorem:

Theorem 3.1 Up to terms of order n^{-1} ,

$$\text{MSE}(d_1) \geq \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 B_{(\alpha)},$$

with the equality holding if $f_1(1, 1) = A$ and $f_2(1, 1) = B$.

A family of estimators wider than d_1 is defined by

$$d_2 = F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)}), \quad (3.8)$$

where the function $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)})$ satisfies $F(D) = R_{(\alpha)}$ and $F_1(D) = 1$.

To the first degree of approximation the bias and MSE of d_2 are, respectively, given by

$$B(d_2) = B(\hat{R}_{\alpha(n-r)}) + \frac{\theta^*}{2} [C_2^2 F_{22}(D) + (\lambda_{004} - 1) F_{33}(D) + 2R_{(\alpha)} K_{(\alpha)} F_{12}(D) + 2R_{(\alpha)} d_{(\alpha)} F_{13}(D) + 2\lambda_{003} C_2 F_{23}(D)] \quad (3.9)$$

and

$$\begin{aligned} \text{MSE}(d_2) &= \text{MSE}(\hat{R}_{\alpha(n-r)}) \\ &+ \theta^* [C_2^2 F_2^2(D) + (\lambda_{004} - 1) F_3^2(D) + 2R_{(\alpha)} K_{(\alpha)} C_2^2 F_{12}(D) \\ &+ 2R_{(\alpha)} d_{(\alpha)} F_{13}(D) + 2\lambda_{003} C_2 F_{23}(D)]. \end{aligned} \quad (3.10)$$

The $\text{MSE}(d_2)$ is minimized for

$$F_2(D) = R_{(\alpha)} A = A^* \text{ (say)} \quad \text{and} \quad F_3(D) = R_{(\alpha)} B = B^* \text{ (say)}. \quad (3.11)$$

Thus, the minimum MSE of d_2 is given by

$$\text{MSE}(d_2) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 B_{(\alpha)} = \min. \text{MSE}(d_1). \quad (3.12)$$

It follows from (3.12) that to the first degree of approximation the minimum MSE of d_2 is equal to that of d_1 (i.e. $\min. \text{MSE}(d_2) = \min. \text{MSE}(d_1)$) and is not reduced.

Furthermore, we note that the estimators of the type

$$d_{2(1)} = \hat{R}_{\alpha(n-r)} + \alpha_1(u_{(n-r)} - 1), \quad d_{2(2)} = \frac{\bar{y}_{0(n-r)} + \alpha_1(u_{(n-r)} - 1)}{\bar{y}_{1(n-r)} + \alpha_2(u_{(n-r)} - 1)}$$

are members of the family d_2 but not of d_1 , where α_i , $i = 1, 2$, are real constants.

Remark 3.1 The following families of estimators of $R_{(\alpha)}$:

$$d_{1(1)} = \hat{R}_{\alpha(n-r)} f(u_{(n-r)}) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}) \quad (3.13)$$

and

$$d_{1(2)} = \hat{R}_{\alpha(n-r)} f(v_{(n-r)}) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.14)$$

are members of the family d_1 , where $f(u_{(n-r)})$ and $f(v_{(n-r)})$ are the functions of $u_{(n-r)}$ and $v_{(n-r)}$, respectively, such that $f(1) = 1$. The bias of $d_{1(1)}$ and $d_{1(2)}$ can be obtained from (3.2) just by putting $[f_1(1, 1) = f_1(1), f_{11}(1, 1) = f_{11}(1), f_2(1, 1) = f_{12}(1, 1) = f_{22}(1, 1) = 0]$ and $[f_2(1, 1) = f_2(1), f_{22}(1, 1) = f_{22}(1), \text{ and } f_1(1, 1) = f_{12}(1, 1) = f_{11}(1, 1) = 0]$. The MSEs of $d_{1(1)}$ and $d_{1(2)}$ can be obtained from (3.3) by putting $[f_1(1, 1) = f_1(1), f_2(1) = 1]$ and $[f_2(1, 1) = f_2(1), f_1(1) = 1]$. The MSEs of $d_{1(1)}$ and $d_{1(2)}$ are, respectively, minimized for

$$f_1(1) = -K_{(\alpha)} \quad (3.15)$$

and

$$f_2(1) = -d_{(\alpha)}/(\lambda_{004} - 1), \quad (3.16)$$

which give the minimum MSEs of $d_{1(1)}$ and $d_{1(2)}$, respectively, as

$$\min.\text{MSE}(d_{1(1)}) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2 \quad (3.17)$$

and

$$\min.\text{MSE}(d_{1(2)}) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 d_{(\alpha)}^2 / (\lambda_{004} - 1). \quad (3.18)$$

Remark 3.2 The following are the families of estimators of $R_{(\alpha)}$:

$$d_{2(3)} = F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}), \quad (3.19)$$

and

$$d_{2(4)} = F(\hat{R}_{\alpha(n-r)}, v_{(n-r)}) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.20)$$

where $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)})$ and $F(\hat{R}_{\alpha(n-r)}, v_{(n-r)})$ are the functions of $(\hat{R}_{\alpha(n-r)}, u_{(n-r)})$ and $(\hat{R}_{\alpha(n-r)}, v_{(n-r)})$ such that $F(R_{(\alpha)}, 1) = R_{(\alpha)}$ and $F_1(R_{(\alpha)}, 1) = 1$.

The biases of $d_{2(3)}$ and $d_{2(4)}$ can be obtained from (3.9) by putting $[F_{22}(D) = F_{22}(R_{(\alpha)}, 1), F_{12}(D) = F_{12}(R_{(\alpha)}, 1), F_{13}(D) = F_{23}(D) = F_{33}(D) = 0]$ and $[F_{33}(D) = F_{33}(R_{(\alpha)}, 1), F_{13}(D) = F_{13}(R_{(\alpha)}, 1), F_{22}(D) = F_{12}(D) = F_{23}(D) = 0]$, respectively. The MSEs of $d_{2(3)}$ and $d_{2(4)}$ can be obtained from (3.10) by putting $[F_2(D) = F_2(R_{(\alpha)}, 1), F_3(D) = 0]$ and $[F_3(D) = F_3(R_{(\alpha)}, 1), F_2(D) = 0]$, respectively. The MSEs of $d_{2(3)}$ and $d_{2(4)}$ are, respectively, minimized for

$$F_2(R_{(\alpha)}, 1) = -R_{(\alpha)} K_{(\alpha)} \quad (3.21)$$

and

$$F_3(R_{(\alpha)}, 1) = -R_{(\alpha)} d_{(\alpha)} / (\lambda_{004} - 1). \quad (3.22)$$

Thus, the resulting minimum MSEs of $d_{2(3)}$ and $d_{2(4)}$ are, respectively, given by

$$\min.\text{MSE}(d_{1(1)}) = \min.\text{MSE}(d_{2(3)}) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2 \quad (3.23)$$

and

$$\begin{aligned} \min.\text{MSE}(d_{1(2)}) &= \min.\text{MSE}(d_{2(4)}) \\ &= \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 d_{(\alpha)}^2 / (\lambda_{004} - 1). \end{aligned} \quad (3.24)$$

Now we state the following corollary:

Corollary 3.1 Up to terms of order n^{-1} ,

- (i) $\text{MSE}(d_{1(1)} \text{ or } d_{2(3)}) \geq [\text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2]$,
with the equality holding if $f_1(1) = -K_{(\alpha)}$ for $d_{1(1)}$, and if $F_2(R_{(\alpha)}, 1) = -R_{(\alpha)} K_{(\alpha)}$ for $d_{2(3)}$.
- (ii) $\text{MSE}(d_{1(2)} \text{ or } d_{2(4)}) \geq [\text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 d_{(\alpha)}^2 / (\lambda_{004} - 1)]$,
with the equality holding if $f_2(1) = -d_{(\alpha)} / (\lambda_{004} - 1)$, and if $F_3(R_{(\alpha)}, 1) = -R_{(\alpha)} d_{(\alpha)} / (\lambda_{004} - 1)$ for $d_{2(4)}$.

Remark 3.3 It is to be mentioned here that all the families of estimators $d_1, d_{1(1)}, d_{1(2)}, d_{2(3)}$ and $d_{2(4)}$ are members of the family d_2 . The biases and MSEs of these estimators can be obtained from (3.9) and (3.10), respectively, by putting the suitable values of the derivatives, as per Table 2.

Remark 3.4 If we set $\alpha = 0$ in (3.1) and (3.8), we get the families of estimators of the population mean $\bar{Y}_0$ as

$$d_1^{(1)} = \bar{y}_{0(n-r)} f(u_{(n-r)}, v_{(n-r)}) \quad (3.25)$$

and

$$d_2^{(1)} = F(\bar{y}_{0(n-r)}, u_{(n-r)}, v_{(n-r)}), \quad (3.26)$$

where $f(u_{(n-r)}, v_{(n-r)})$ and $F(\bar{y}_{0(n-r)}, u_{(n-r)}, v_{(n-r)})$ are the functions of $(u_{(n-r)}, v_{(n-r)})$ and $(\bar{y}_{0(n-r)}, u_{(n-r)}, v_{(n-r)})$, respectively, such that $f(1, 1) = 1$ and $F(D_0) = \bar{Y}_0$ implies that $F_1(D_0) = 1$. The biases and MSEs of $d_1^{(1)}$ and $d_2^{(1)}$ can be obtained from ((3.2), (3.3)) and ((3.9), (3.10)), respectively, by putting $\alpha = 0$. The optimum values of $(f_1(1, 1), f_2(1, 1))$ and $(F_2(D_0), F_3(D_0))$ for which the MSEs of $d_1^{(1)}$ and $d_2^{(1)}$ are minimum, and the minimum MSEs of $d_1^{(1)}$ and $d_2^{(1)}$ can be obtained

Table 2 Values of the derivatives

Estimator	$F_2(D)$	$F_3(D)$	$F_{12}(D)$	$F_{13}(D)$
d_1	$R_{(\alpha)} f_1(1, 1)$	$R_{(\alpha)} f_2(1, 1)$	$f_1(1, 1)$	$f_2(1, 1)$
$d_{1(1)}$	$R_{(\alpha)} f_1(1)$	0	$f_1(1)$	0
$d_{1(2)}$	0	$R_{(\alpha)} f_2(1)$	0	$f_2(1)$
$d_{2(3)}$	$F_2(R_{(\alpha)}, 1)$	0	$F_{12}(R_{(\alpha)}, 1)$	0
$d_{2(4)}$	0	$F_3(R_{(\alpha)}, 1)$	0	$F_{13}(R_{(\alpha)}, 1)$
Estimator	$F_{22}(D)$	$F_{23}(D)$	$F_{33}(D)$	
d_1	$R_{(\alpha)} f_{11}(1, 1)$	$R_{(\alpha)} f_{12}(1, 1)$	$R_{(\alpha)} f_{22}(1, 1)$	
$d_{1(1)}$	$R_{(\alpha)} f_{11}(1)$	0	0	
$d_{1(2)}$	0	0	$R_{(\alpha)} f_{22}(1)$	
$d_{2(3)}$	$F_{22}(R_{(\alpha)}, 1)$	0	0	
$d_{2(4)}$	0	0	$F_{33}(R_{(\alpha)}, 1)$	

Table 3 Values of the derivatives

Estimator	$F_2(D)$	$F_3(D)$	$F_{12}(D)$	$F_{13}(D)$
$d_1^{(1)}$	$\bar{Y}_0 f_1(1, 1)$	$\bar{Y}_0 f_2(1, 1)$	$f_1(1, 1)$	$f_2(1, 1)$
$d_2^{(1)}$	$F_2(D_0)$	$F_3(D_0)$	$F_{12}(D_0)$	$F_{13}(D_0)$
$d_{1(1)}^{(1)}$	$\bar{Y}_0 f_1(1)$	0	$f_1(1)$	0
$d_{1(2)}^{(1)}$	0	$\bar{Y}_0 f_2(1)$	0	$f_2(1)$
$d_{2(1)}^{(1)}$	$F_2(\bar{Y}_0, 1)$	0	$F_{12}(\bar{Y}_0, 1)$	0
$d_{2(2)}^{(1)}$	0	$F_3(\bar{Y}_0, 1)$	0	$F_{13}(\bar{Y}_0, 1)$
Estimator	$F_{22}(D)$	$F_{23}(D)$	$F_{33}(D)$	
$d_1^{(1)}$	$\bar{Y}_0 f_{11}(1, 1)$	$\bar{Y}_0 f_{12}(1, 1)$	$\bar{Y}_0 f_{22}(1, 1)$	
$d_2^{(1)}$	$F_{22}(D_0)$	$F_{23}(D_0)$	$F_{33}(D_0)$	
$d_{1(1)}^{(1)}$	$\bar{Y}_0 f_{11}(1)$	0	0	
$d_{1(2)}^{(1)}$	0	0	$\bar{Y}_0 f_{22}(1)$	
$d_{2(1)}^{(1)}$	$F_{22}(\bar{Y}_0, 1)$	0	0	
$d_{2(2)}^{(1)}$	0	0	$F_{33}(\bar{Y}_0, 1)$	

from ((3.6), (3.11)) and ((3.7), (3.12)), respectively, by putting $\alpha = 0$. The common min.MSE of $d_1^{(1)}$ and $d_2^{(1)}$ is

$$\begin{aligned} \min.\text{MSE}(d_1^{(1)}) &= \min.\text{MSE}(d_2^{(1)}) \\ &= \theta^* S_0^2 [1 - \rho_{02}^2 - (\lambda_{003}\rho_{02} - \lambda_{102})^2 / \Delta]. \end{aligned} \quad (3.27)$$

Thus, by putting $\alpha = 0$ in Theorem 3.1, we get the following corollaries:

Corollary 3.2 Up to terms of order n^{-1} ,

- (i) $\text{MSE}(d_1^{(1)}) \geq \theta^* S_0^2 [1 - \rho_{02}^2 - (\lambda_{003}\rho_{02} - \lambda_{102})^2 / \Delta]$,
with the equality holding if $f_1(1, 1) = A_{11}$ and $f_2(1, 1) = B_{11}$,
- (ii) $\text{MSE}(d_2^{(1)}) \geq \theta^* S_0^2 [1 - \rho_{02}^2 - (\lambda_{003}\rho_{02} - \lambda_{102})^2 / \Delta]$,
with the equality holding if $F_2(D_0) = \bar{Y}_0 A_{11}$ and $F_3(D_0) = \bar{Y}_0 B_{11}$, where A_{11} and B_{11} can be derived from A and B , respectively, by putting $\alpha = 0$.

Remark 3.5 The following classes are estimators of the population mean $\bar{Y}_0$:

$$d_{1(1)}^1 = \bar{y}_{0(n-r)} f(u_{(n-r)}) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}), \quad (3.28)$$

$$d_{1(2)}^1 = \bar{y}_{0(n-r)} f(v_{(n-r)}) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.29)$$

$$d_{2(1)}^1 = F(\bar{y}_{0(n-r)}, u_{(n-r)}) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}), \quad (3.30)$$

$$d_{2(2)}^1 = F(\bar{y}_{0(n-r)}, v_{(n-r)}) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.31)$$

where $f(u_{(n-r)})$, $f(v_{(n-r)})$, $F(\bar{y}_{0(n-r)}, u_{(n-r)})$ and $F(\bar{y}_{0(n-r)}, v_{(n-r)})$ are the functions of $u_{(n-r)}$, $v_{(n-r)}$, and $(\bar{y}_{0(n-r)}, u_{(n-r)})$ and $(\bar{y}_{0(n-r)}, v_{(n-r)})$, respectively, such that $f(1) = 1$ and $F(\bar{Y}_0, 1) = \bar{Y}_0$ which implies that $F_1(\bar{Y}_0, 1) = 1$. It is to be noted that (i) $d_{1(1)}^{(1)}$ and $d_{1(2)}^{(1)}$, and (ii) $d_{2(1)}^{(1)}$ and $d_{2(2)}^{(1)}$, are members of d_1 and d_2 , respectively. Thus, the biases and MSEs of these estimators can be obtained from (3.9) and (3.10), respectively, by putting $\alpha = 0$ and the suitable values of the derivatives from Table 3.

The optimum values of the derivatives, for which the MSEs of $d_{1(1)}^{(1)}$, $d_{1(2)}^{(1)}$, $d_{2(1)}^{(1)}$, and $d_{2(2)}^{(1)}$ are minimum, and the minimum MSEs of these estimators can be obtained from (3.15), (3.16), (3.17), (3.18), (3.21), (3.22), (3.23), and (3.24) just by putting $\alpha = 0$. The minimum MSEs of $d_{1(1)}^{(1)}$, $d_{1(2)}^{(1)}$, $d_{2(1)}^{(1)}$, and $d_{2(2)}^{(1)}$ are, respectively, given by

$$\min.\text{MSE}(d_{1(1)}^{(1)}) = \min.\text{MSE}(d_{2(1)}^{(1)}) = \theta^* S_0^2 (1 - \rho_{02}^2) \quad (3.32)$$

and

$$\min.\text{MSE}(d_{1(2)}^{(1)}) = \min.\text{MSE}(d_{2(2)}^{(1)}) = \theta^* S_0^2 (1 - \lambda_{102}^2 / (\lambda_{004} - 1)). \quad (3.33)$$

Thus, we state the following corollary by putting $\alpha = 0$ in Corollary 3.1.

Corollary 3.3 Up to terms of order n^{-1} ,

- (i) $\text{MSE}(d_{1(1)}^{(1)} \text{ or } d_{2(1)}^{(1)}) \geq \theta^* S_0^2 (1 - \rho_{02}^2)$,
with the equality holding if $f_1(1) = -K_{02}$ for $d_{1(1)}^{(1)}$, and if $F_2(\bar{Y}_0, 1) = -\bar{Y}_0 K_{02}$ for $d_{2(1)}^{(1)}$.
- (ii) $\text{MSE}(d_{1(2)}^{(1)} \text{ or } d_{2(2)}^{(1)}) \geq \theta^* S_0^2 (1 - \lambda_{102}^2 / (\lambda_{004} - 1))$,
with the equality holding if $f_2(1) = -\lambda_{102} C_0 / (\lambda_{004} - 1)$ for $d_{1(2)}^{(1)}$, and if $F_3(\bar{Y}_0, 1) = -\bar{Y}_0 \lambda_{102} C_0 / (\lambda_{004} - 1)$ for $d_{2(2)}^{(1)}$.

It is to be noted that the estimator

$$d_{2(3)}^{(1)} = \bar{y}_{0(n-r)} - \alpha_1 \bar{Y}_2(u_{(n-r)} - 1),$$

suggested by Singh et al. (2000), is a particular member of the family $d_{2(1)}^{(1)}$ but not of $d_{1(1)}^{(1)}$, where α_1 is a constant. The minimum MSE of $d_{2(3)}^{(1)}$ is the same as given by (3.32).

3.1.1 Estimators with estimated optimum values

It is to be noted that the optimum values of the parameters involved in estimators depend on unknown population parameters such as λ_{003} , λ_{004} , λ_{102} , λ_{012} , C_2 , K_{01} , K_{02} , ..., etc. Thus, to use such an estimator one has to use guessed or estimated values of these parameters. Guessed values of population parameters can be obtained from either past data or experience. If the guessed values are not known then it is advisable to use sample data at hand to estimate these parameters. The estimated optimum values of $f_1(1, 1)$ and $f_2(1, 1)$ are given by

$$\hat{f}_1(1, 1) = \hat{A} \quad \text{and} \quad \hat{f}_2(1, 1) = \hat{B}. \quad (3.34)$$

In case of estimated optimum values we find the MSE as follows: from the regularity conditions, defined in the [Appendix](#), for d_1 we desire a function $f(u_{(n-r)}, v_{(n-r)})$ such that $f(1, 1) = 1$, $f_1(1, 1) = A$, $f_2(1, 1) = B$, which indicates that the function $f(u_{(n-r)}, v_{(n-r)})$ will contain not only $u_{(n-r)}$, $v_{(n-r)}$ but A and B as well, and, thus, we need a function $f^*(u_{(n-r)}, v_{(n-r)}, A, B)$ such that $f^*(Z) = 1$, $f_1^*(Z) = A$, $f_2^*(Z) = B$. Since in such function $f^*(u_{(n-r)}, v_{(n-r)}, A, B)$ so required, A and B are unknown, we may take $f^{**}(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B}) = f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ (replacing A, B by their estimated values). Now $f^*(Z) = 1$, $f_1^*(Z) = A$, $f_2^*(Z) = B$.

We may consider

$$d_1^* = \hat{R}_{\alpha(n-r)} f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B}) \quad (3.35)$$

as an estimator of $R_{(\alpha)}$. Expanding $f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ about the point Z into Taylor's series, we have

$$\begin{aligned} d_1^* &= \hat{R}_{\alpha(n-r)} [1 + (u_{(n-r)} - 1)f_1^*(Z) + (v_{(n-r)} - 1)f_2^*(Z) \\ &\quad + (\hat{A} - A)f_3^*(Z) + (\hat{B} - B)f_4^*(Z) + \dots]. \end{aligned}$$

Then in terms of e_i 's we have

$$\begin{aligned} d_1^* - \hat{R}_{(\alpha)} &= R_{\alpha} [e_0 - \alpha e_1 + e_2 A + e_4 B + (e_0 - \alpha e_1)e_2 A + (e_0 - \alpha e_1)e_4 B \\ &\quad + (\hat{A} - A)f_3^*(Z) + (\hat{B} - B)f_4^*(Z) + \dots]. \end{aligned} \quad (3.36)$$

Squaring both sides of (3.36) and taking expectation, we have the MSE of d_1^* equal to the minimum MSE of d_1 in (3.7) if $f_3^*(Z) = 0$, $f_4^*(Z) = 0$. Thus, if we consider the family of estimators (depending upon estimated optimum values)

$$d_1^* = \hat{R}_{\alpha(n-r)} f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B}) \quad (3.37)$$

such that $f^*(Z) = 1$, $f_1^*(Z) = A$, $f_2^*(Z) = B$, $f_3^*(Z) = 0$ and $f_4^*(Z) = 0$ such an estimator d_1^* attains the min.MSE(d_1), given in (3.7).

Now we state the following theorems. Proofs can also be obtained by following Randles (1982).

Theorem 3.2 *The family of estimators (based on estimated optimum values) of $R_{(\alpha)}$, defined by*

$$d_1^* = \hat{R}_{\alpha(n-r)} f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B}),$$

has the MSE to the first degree of approximation equal to the minimum MSE of d_1 , that is,

$$\text{MSE}(d_1^*) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 B_{(\alpha)} = \text{min.MSE}(d_1),$$

where $f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ is a function of $(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ such that $f^*(Z) = 1$, $f_1^*(Z) = A$, $f_2^*(Z) = B$, $f_3^*(Z) = 0$, and $f_4^*(Z) = 0$.

Theorem 3.3 A wider family of estimators (based on estimated optimum values) of $R_{(\alpha)}$, defined by $d_2^* = F^*(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$, has the MSE to the first degree of approximation the same as the min.MSE of d_2 , given by (3.12), i.e.,

$$\text{MSE}(d_2^*) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta^* R_{(\alpha)}^2 B_{(\alpha)} = \min.\text{MSE}(d_2),$$

where $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ is a function of $(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ such that $F^*(Q^*) = R_{(\alpha)}$, $F_1^*(Q^*) = 1$, $F_2^*(Q^*) = A^*$, $F_3^*(Q^*) = B^*$, $F_4^*(Q^*) = 0$ and $F_5^*(Q^*) = 0$.

In similar fashion many other families of estimators of the ratio $R_{(\alpha)}$ based on estimated optimum values can be derived. Furthermore, we state the following theorem:

Theorem 3.4 Estimators of the minimum MSE of d_1 (or d_2) or MSE of d_1^* (or d_2^*), $d_1^{(1)}$ (or $d_2^{(1)}$), and $d_{1(1)}^{(1)}$ (or $d_{2(1)}^{(1)}$) are, respectively, given by

$$\begin{aligned} \min.\widehat{\text{MSE}}(d_1) &= \min.\widehat{\text{MSE}}(d_2) = \widehat{\text{MSE}}(d_2^*) \\ &= \widehat{\text{MSE}}(\hat{R}_{\alpha(n-r)}) - \hat{\theta}^{*2} \hat{R}_{\alpha(n-r)}^2 \left\{ \hat{R}_{\alpha(n-r)}^2 \hat{C}_{2(n-r)}^2 - \frac{\hat{\Delta}_{2(n-r)}^*}{\hat{\Delta}_{(n-r)}} \right\}, \\ \min.\widehat{\text{MSE}}(d_1^{(1)}) &= \min.\widehat{\text{MSE}}(d_2^{(1)}) \\ &= \hat{\theta}^* s_{0(n-r)}^2 \left[1 - \hat{\rho}_{02(n-r)}^2 - \frac{\{\hat{\lambda}_{003(n-r)} \hat{\rho}_{02(n-r)} - \hat{\lambda}_{102(n-r)}\}^2}{\hat{\Delta}_{(n-r)}} \right], \end{aligned}$$

and

$$\begin{aligned} \min.\widehat{\text{MSE}}(d_{1(1)}^{(1)}) &= \min.\widehat{\text{MSE}}(d_{2(1)}^{(1)}) \\ &= \hat{\theta}^* s_{0(n-r)}^2 (1 - \hat{\rho}_{02(n-r)}^2), \end{aligned}$$

where

$$\widehat{\text{MSE}}(\hat{R}_{\alpha(n-r)}) = \hat{\theta}^* \hat{R}_{\alpha(n-r)}^2 [\hat{C}_{0(n-r)}^2 + \alpha \hat{C}_{2(n-r)}^2 \{\alpha - \hat{K}_{01(n-r)}\}].$$

3.2 Strategy II

We consider the situation when information on variables y_0 and y_1 cannot be obtained for r units, while the population mean $\bar{Y}_2$ and the variance S_2^2 of the auxiliary variable y_2 are known. We propose the following family of estimators of $R_{(\alpha)}$

$$d_3 = \hat{R}_{\alpha(n-r)} \phi(u, v), \quad (3.38)$$

where $\phi(u, v)$ is a function of (u, v) such that $\phi(1, 1) = 1$ and satisfying certain conditions similar to those for $f(\cdot)$ in d_1 . Up to terms of order $O(n^{-1})$, the bias and

MSE of d_3 are, respectively, given by

$$B(d_3) = B(\hat{R}_{\alpha(n-r)}) + \frac{\theta R_{(\alpha)}}{2} [2K_{(\alpha)} C_2^2 \phi_1(1, 1) + 2d_{(\alpha)} \phi_2(1, 1) + C_2^2 \phi_{11}(1, 1) + (\lambda_{004} - 1) \phi_{22}(1, 1) + 2\lambda_{003} C_2 \phi_{12}(1, 1)] \quad (3.39)$$

and

$$\begin{aligned} \text{MSE}(d_3) &= \text{MSE}(\hat{R}_{\alpha(n-r)}) \\ &+ \theta R_{(\alpha)}^2 [C_2^2 \phi_1^2(1, 1) + (\lambda_{004} - 1) \phi_2^2(1, 1) + 2\lambda_{003} C_2 \phi_1(1, 1) \phi_2(1, 1) \\ &+ K_{(\alpha)} C_2^2 \phi_1(1, 1) + 2d_{(\alpha)} \phi_2(1, 1)]. \end{aligned} \quad (3.40)$$

The bias and MSE of an estimator belonging to the proposed family d_3 can be easily obtained from (3.2) and (3.3), respectively. The MSE of d_3 is minimized for

$$\phi_1(1, 1) = A \quad \text{and} \quad \phi_2(1, 1) = B. \quad (3.41)$$

Thus, the resulting minimum MSE of d_3 is given by

$$\min.\text{MSE}(d_3) = \text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta R_{(\alpha)}^2 B_{(\alpha)}, \quad (3.42)$$

where $\text{MSE}(\hat{R}_{\alpha(n-r)})$ is given by (3.5). Thus, we state the following theorem:

Theorem 3.5 Up to term of order n^{-1} ,

$$\text{MSE}(d_3) \geq [\text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta R_{(\alpha)}^2 B_{(\alpha)}],$$

with the equality holding if $\phi_1(1, 1) = A$ and $\phi_2(1, 1) = B$.

A family of estimators wider than d_3 is defined by

$$d_4 = \Phi(\hat{R}_{\alpha(n-r)}, u, v), \quad (3.43)$$

where $\Phi(\hat{R}_{\alpha(n-r)}, u, v)$ is a function of $(\hat{R}_{\alpha(n-r)}, u, v)$ such that $\Phi(D) = R_{(\alpha)}$ and $\Phi_1(D) = 1$. The bias and MSE of d_4 to the first degree of approximation are, respectively, given by

$$\begin{aligned} B(d_4) &= B(\hat{R}_{\alpha(n-r)}) + \frac{\theta}{2} [2R_{(\alpha)} K_{(\alpha)} C_2^2 \Phi_{12}(D) + 2R_{(\alpha)} d_{(\alpha)} \Phi_{13} \\ &+ C_2^2 \Phi_{22}(D) + (\lambda_{004} - 1) \Phi_{33}(D) + 2\lambda_{003} C_2 \Phi_{23}(D)] \end{aligned} \quad (3.44)$$

and

$$\begin{aligned} \text{MSE}(d_4) &= \text{MSE}(\hat{R}_{\alpha(n-r)}) + \theta [C_2^2 \Phi_2^2(D) + (\lambda_{004} - 1) \Phi_3^2(D) \\ &+ 2\lambda_{003} C_2 \Phi_2(D) \Phi_3(D) + 2R_{(\alpha)} K_{(\alpha)} C_2^2 \Phi_2(D) \\ &+ 2R_{(\alpha)} d_{(\alpha)} \Phi_3(D)]. \end{aligned} \quad (3.45)$$

The MSE of d_4 is minimized for

$$\Phi_2(D) = R_{(\alpha)}A = A^* \text{ (say)} \quad \text{and} \quad \Phi_3(D) = R_{(\alpha)}B = B^* \text{ (say)}, \quad (3.46)$$

and the resulting minimum MSE of d_4 is given by

$$\min.\text{MSE}(d_4) = [\text{MSE}(\hat{R}_{\alpha(n-r)}) - \theta R_{(\alpha)}^2 B_{(\alpha)}]. \quad (3.47)$$

It follows from (3.47) that to the first degree of approximation the minimum MSE of d_4 is equal to that of d_3 (i.e., $\min.\text{MSE}(d_4) = \min.\text{MSE}(d_3)$) and is not reduced. The difference type estimator, $d_{4(1)} = \hat{R}_{\alpha(n-r)} + \alpha_1(u - 1) + \alpha_2(v - 1)$, is a member of the family d_4 , but not of d_3 .

Remark 3.6 The following are classes of estimators of $R_{(\alpha)}$:

$$d_{4(1)}^{(1)} = \hat{R}_{\alpha(n-r)}\phi(u) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}), \quad (3.48)$$

$$d_{4(2)}^{(1)} = \hat{R}_{\alpha(n-r)}\phi(v) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.49)$$

$$d_{4(3)}^{(1)} = \Phi(\hat{R}_{\alpha(n-r)}, u) \quad (\text{only } \bar{Y}_2 \text{ of } y_2 \text{ is known}), \quad (3.50)$$

$$d_{4(4)}^{(1)} = \Phi(\hat{R}_{\alpha(n-r)}, v) \quad (\text{only } S_2^2 \text{ of } y_2 \text{ is known}), \quad (3.51)$$

where $\phi(u)$, $\phi(v)$, $\phi(\hat{R}_{\alpha(n-r)}, u)$ and $\phi(\hat{R}_{\alpha(n-r)}, v)$ are the functions of u , v , $(\hat{R}_{\alpha(n-r)}, u)$ and $(\hat{R}_{\alpha(n-r)}, v)$, respectively, such that $\phi(1) = 1$ and $\phi(R_{(\alpha)}, 1) = R_{(\alpha)}$ which implies $\phi_1(R_{(\alpha)}, 1) = 1$. The biases and MSEs of these classes of estimators can be easily obtained from (3.44) and (3.45), respectively, for suitable values of the derivatives.

Remark 3.7 The following are families of estimators of the population mean $\bar{Y}_0$:

$$d_{4(1)}^{(0)} = \bar{y}_{0(n-r)}\phi(u, v), \quad (3.52)$$

$$d_{4(2)}^{(0)} = \Phi(\bar{y}_{0(n-r)}, u, v), \quad (3.53)$$

$$d_{4(3)}^{(0)} = \bar{y}_{0(n-r)}\phi(u), \quad (3.54)$$

$$d_{4(4)}^{(0)} = \bar{y}_{0(n-r)}\phi(v), \quad (3.55)$$

$$d_{4(5)}^{(0)} = \Phi(\bar{y}_{0(n-r)}, u), \quad (3.56)$$

and

$$d_{4(6)}^{(0)} = \Phi(\bar{y}_{0(n-r)}, v), \quad (3.57)$$

where $\phi(u, v)$, $\Phi(\bar{y}_{0(n-r)}, u, v)$, $\phi(u)$, $\phi(v)$, $\Phi(\bar{y}_{0(n-r)}, u)$, and $\Phi(\bar{y}_{0(n-r)}, v)$ are the functions of (u, v) , $(\bar{y}_{0(n-r)}, u, v)$, (u) , (v) , $(\bar{y}_{0(n-r)}, u)$, and $(\bar{y}_{0(n-r)}, v)$ such that $\phi(1, 1) = 1$, $\phi(1) = 1$, $\Phi(\bar{Y}_0, 1) = \bar{Y}_0$ which implies that $\Phi_1(\bar{Y}_0, 1) = 1$, and $\Phi(D_0) = \bar{Y}_0$ which implies $\Phi(D_0) = 1$. The biases and mean squared errors of these

estimators can be obtained from (3.44) and (3.45) by putting $\alpha = 0$ and taking suitable values of the derivatives. The minimum MSE of $d_{4(1)}^{(0)}$ (or $d_{4(2)}^{(0)}$), $d_{4(3)}^{(0)}$ (or $d_{4(5)}^{(0)}$), and $d_{4(4)}^{(0)}$ (or $d_{4(6)}^{(0)}$), respectively, are

$$\begin{aligned} \min.\text{MSE}(d_{4(1)}^{(0)}) &= \min.\text{MSE}(d_{4(2)}^{(0)}) \\ &= \theta S_0^2 \left[1 - \rho_{02}^2 - \frac{(\lambda_{003}\rho_{02} - \lambda_{102})^2}{\Delta} \right] + (\theta^* - \theta)S_0^2, \end{aligned} \quad (3.58)$$

$$\min.\text{MSE}(d_{4(3)}^{(0)}) = \min.\text{MSE}(d_{4(5)}^{(0)}) = \theta S_0^2 (1 - \rho_{02}^2) + (\theta^* - \theta)S_0^2, \quad (3.59)$$

and

$$\begin{aligned} \min.\text{MSE}(d_{4(4)}^{(0)}) &= \min.\text{MSE}(d_{4(6)}^{(0)}) \\ &= \theta S_0^2 \left[1 - \frac{\lambda_{102}^2}{(\lambda_{004} - 1)} \right] + (\theta^* - \theta)S_0^2. \end{aligned} \quad (3.60)$$

Theorem 3.6 Up to terms of order n^{-1} ,

- (i) $\text{MSE}(d_{4(1)}^{(0)} \text{ or } d_{4(2)}^{(0)}) \geq \theta S_0^2 \left[1 - \rho_{02}^2 - \frac{(\lambda_{003}\rho_{02} - \lambda_{102})^2}{\Delta} \right] + (\theta^* - \theta)S_0^2$,
with the equality holding if $\phi_1(1, 1) = A_{11}$ and $\phi_2(1, 1) = B_{11}$ for $d_{4(1)}^{(0)}$, and if $\Phi_2(D_0) = \bar{Y}_0 A_{11}$ and $\Phi_3(D_0) = \bar{Y}_0 B_{11}$ for $d_{4(2)}^{(0)}$.
- (ii) $\text{MSE}(d_{4(3)}^{(0)} \text{ or } d_{4(5)}^{(0)}) \geq \theta S_0^2 (1 - \rho_{02}^2) + (\theta^* - \theta)S_0^2$,
with the equality holding if $\phi_1(1) = -K_{02}$ for $d_{4(3)}^{(0)}$, and if $\Phi_2(\bar{Y}_0, 1) = -K_{02}\bar{Y}_0$ for $d_{4(5)}^{(0)}$.
- (iii) $\text{MSE}(d_{4(4)}^{(0)} \text{ or } d_{4(6)}^{(0)}) \geq \theta S_0^2 \left[1 - \frac{\lambda_{102}^2}{(\lambda_{004} - 1)} \right] + (\theta^* - \theta)S_0^2$,
with the equality holding if $\phi_2(1) = -\lambda_{102}C_0/(\lambda_{004} - 1)$ for $d_{4(4)}^{(0)}$, and if $\phi_3(\bar{Y}_0, 1) = -\lambda_{102}C_0\bar{Y}_0/(\lambda_{004} - 1)$ for $d_{4(6)}^{(0)}$.
It is to be noted that the estimator

$$d_{4(7)}^{(0)} = \bar{y}_{0(n-r)} - \alpha_2 \bar{Y}_2(u - 1)$$

of the population mean $\bar{Y}_0$, suggested by Singh et al. (2000), is a member of the family $d_{4(5)}^{(0)}$ (or $d_{4(2)}^{(0)}$) in (3.56) (or (3.53)) but not of $d_{4(3)}^{(0)}$ in (3.54), where α_2 is a constant. The minimum MSE of $d_{4(7)}^{(0)}$ is the same as given by (3.59).

3.2.1 Estimators with estimated optimum values

The consistent estimators of optimum values $\phi_1(1, 1)$ and $\phi_2(1, 1)$, based on sample data at hand, are

$$\hat{\phi}_1(1, 1) = \hat{A}_1 \text{ (say)} \quad \text{and} \quad \hat{\phi}_2(1, 1) = \hat{B}_1 \text{ (say)}. \quad (3.61)$$

Thus, the resulting family of estimators of $R_{(\alpha)}$ is defined as

$$d_3^* = \hat{R}_{\alpha(n-r)} \phi^*(u, v, \hat{A}_1, \hat{B}_1), \quad (3.62)$$

where $\phi^*(\cdot)$ is a function of $(u, v, \hat{A}_1, \hat{B}_1)$ such that $\phi^*(Z) = 1$, $\phi_1^*(Z) = A$, $\phi_2^*(Z) = B$, $\phi_3^*(Z) = 0$, and $\phi_4^*(Z) = 0$. It can be shown that to the first degree of approximation $\text{MSE}(d_3^*) = \min.\text{MSE}(d_3)$, where $\min.\text{MSE}(d_3)$ is given by (3.42). Furthermore, a family of estimators (based on estimated optimum values) of $R_{(\alpha)}$ wider than d_3^* is defined by

$$d_4^* = \Phi^*(\hat{R}_{\alpha(n-r)}, u, v, \hat{A}_1^*, \hat{B}_1^*),$$

where $\Phi^*(\hat{R}_{\alpha(n-r)}, u, v, \hat{A}_1^*, \hat{B}_1^*)$ is a function of $(\hat{R}_{\alpha(n-r)}, u, v, \hat{A}_1^*, \hat{B}_1^*)$ such that $\Phi^*(Z^*) = R_{(\alpha)}$, $\Phi_1^*(Z^*) = 1$, $\Phi_2^*(\Psi) = \frac{\partial \Phi^*(\bullet)}{\partial u} \Big|_{\psi} = A^*$, $\Phi_3^*(Z^*) = B^*$, $\Phi_4^*(Z^*) = 0$, and $\Phi_5^*(Z^*) = 0$. It can be proved that to the first degree of approximation $\text{MSE}(d_4^*) = \min.\text{MSE}(d_4)$, where $\min.\text{MSE}(d_4)$ is given by (3.47). Similarly, various families of estimators of $R_{(\alpha)}$ based on estimated optimum values can be defined. Consistent estimators for estimating the minimum MSEs of various families are given in the following theorems.

Theorem 3.7 Estimators of the minimum MSE of d_3 (or MSE of d_3^*), $d_{4(1)}^{(0)}$ (or $d_{4(2)}^{(0)}$), $d_{4(3)}^{(0)}$ (or $d_{4(5)}^{(0)}$) and $d_{4(4)}^{(0)}$ (or $d_{4(6)}^{(0)}$) are, respectively, given by

$$\begin{aligned} \min.\widehat{\text{MSE}}(d_3) &= \widehat{\text{MSE}}(d_3^*) \\ &= \widehat{\text{MSE}}(\hat{R}_{\alpha(n-r)}) \\ &\quad - \theta \hat{R}_{\alpha(n-r)}^2 \left\{ \hat{K}_{(\alpha)}^{*2} \hat{C}_2^2 + \frac{(\hat{K}_{(\alpha)}^* \hat{\lambda}_{003} \hat{C}_2 - \hat{d}_{\alpha(n-r)})^2}{\hat{\lambda}_{004} - \hat{\lambda}_{003}^2 - 1} \right\}, \\ \min.\widehat{\text{MSE}}(d_{4(1)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{4(2)}^{(0)}) \\ &= \theta s_{0(n-r)}^2 \left[1 - \hat{\rho}_{02(n-r)}^2 - \frac{(\hat{\lambda}_{003} \hat{\rho}_{02(n-r)} - \hat{\lambda}_{102(n-r)})^2}{\hat{\lambda}_{004} - \hat{\lambda}_{003}^2 - 1} \right] \\ &\quad + \left(\frac{1}{n\hat{q} + 2\hat{p}} - \frac{1}{n} \right) s_{0(n-r)}^2, \\ \min.\widehat{\text{MSE}}(d_{4(3)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{4(5)}^{(0)}) \\ &= \left(\frac{1}{n\hat{q} + 2\hat{p}} - \frac{1}{n} \right) s_{0(n-r)}^2 + \theta s_{0(n-r)}^2 (1 - \hat{\rho}_{02(n-r)}^2), \end{aligned}$$

and

$$\begin{aligned} \min.\widehat{\text{MSE}}(d_{4(4)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{4(6)}^{(0)}) \\ &= \theta s_{0(n-r)}^2 \left\{ 1 - \frac{\hat{\lambda}_{102(n-r)}^2}{(\hat{\lambda}_{004} - 1)} \right\} + \left(\frac{1}{n\hat{q} + 2\hat{p}} - \frac{1}{n} \right) s_{\alpha(n-r)}^2. \end{aligned}$$

3.3 Strategy III

We again consider the situation when information on study variables y_0 and y_1 cannot be obtained for r units while information on the auxiliary variable y_2 is obtained for all the sample units. But the population mean $\bar{Y}_2$ and the variance S_2^2 of the auxiliary character y_2 are not known. Under these circumstances we define the following family of estimators of $R_{(\alpha)}$

$$d_5 = \hat{R}_{\alpha(n-r)} a(u^*, v^*), \quad (3.63)$$

where $a(u^*, v^*)$ is a function of u^* and v^* such that $a(1, 1) = 1$ and satisfying certain regularity conditions, as given in (3.1). To terms of order n^{-1} , the bias and MSE of d_5 are, respectively, given by

$$\begin{aligned} B(d_5) = B(\hat{R}_{\alpha(n-r)}) + (\theta^* - \theta) \frac{R_{(\alpha)}}{2} [C_2^2 a_{11}(1, 1) + (\lambda_{004} - 1) a_{22}(1, 1) \\ + 2\lambda_{003} C_2 a_{12}(1, 1) + 2K_{(\alpha)} C_2^2 a_1(1, 1) + 2d_{(\alpha)} a_2(1, 1)] \end{aligned} \quad (3.64)$$

and

$$\begin{aligned} \text{MSE}(d_5) = \text{MSE}(\hat{R}_{\alpha(n-r)}) + R_{\alpha(n-r)}^2 (\theta^* - \theta) [C_2^2 a_1^2(1, 1) \\ + (\lambda_{004} - 1) a_2^2(1, 1) + 2\lambda_{003} C_2 a_1(1, 1) a_2(1, 1) \\ + 2K_{(\alpha)} C_2^2 a_1(1, 1) + 2d_{(\alpha)} a_2(1, 1)]. \end{aligned} \quad (3.65)$$

The $\text{MSE}(d_5)$ is minimized for

$$a_1(1, 1) = A \quad \text{and} \quad a_2(1, 1) = B. \quad (3.66)$$

Thus, the resulting (minimum) MSE of d_5 is given by

$$\begin{aligned} \min.\text{MSE}(d_5) &= \text{MSE}(\hat{R}_{\alpha(n-r)}) - (\theta^* - \theta) R_{(\alpha)}^2 B_{(\alpha)} \\ &= \min.\text{MSE}(d_1) + \theta R_{(\alpha)}^2 B_{(\alpha)}, \end{aligned} \quad (3.67)$$

where $\min.\text{MSE}(d_1)$ is given in (3.7). Thus, we have the following theorem:

Theorem 3.8 Up to terms of order n^{-1} , $\text{MSE}(d_5) \geq \min.\text{MSE}(d_1) + \theta R_{(\alpha)}^2 B_{(\alpha)}$, with the equality holding if $a_1(1, 1) = A$ and $a_2(1, 1) = B$.

Now we define a wider family of estimators for $R_{(\alpha)}$, namely,

$$d_6 = A(\hat{R}_{\alpha(n-r)}, u^*, v^*), \quad (3.68)$$

where $A(\hat{R}_{\alpha(n-r)}, u^*, v^*)$ is a function of $(\hat{R}_{\alpha(n-r)}, u^*, v^*)$ such that $A(D) = R_{(\alpha)}$ and $A_1(D) = 1$. To the first degree of approximation the bias of MSE of d_6 are,

respectively, given by

$$\begin{aligned} B(d_6) &= B(\hat{R}_{\alpha(n-r)}) \\ &+ \frac{(\theta^* - \theta)}{2} [C_2^2 A_{22}(D) + (\lambda_{004} - 1) A_{33}(D) + 2\lambda_{003} C_2 A_{23}(D) \\ &+ 2R_{(\alpha)} C_2^2 K_{(\alpha)} A_{12}(D) + 2R_{(\alpha)} d_{(\alpha)} A_{13}(D)] \end{aligned} \quad (3.69)$$

and

$$\begin{aligned} \text{MSE}(d_6) &= \text{MSE}(\hat{R}_{\alpha(n-r)}) + (\theta^* - \theta) [C_2^2 A_2^2(D) + (\lambda_{004} - 1) A_3^2(D) \\ &+ 2\lambda_{003} C_2 A_2(D) A_3(D) + 2R_{(\alpha)} C_2^2 K_{(\alpha)} A_2(D) \\ &+ 2R_{(\alpha)} d_{(\alpha)} A_3(D)]. \end{aligned} \quad (3.70)$$

The $\text{MSE}(d_6)$ is minimized for

$$A_2(D) = R_{(\alpha)} A = A^* \text{ and } A_3(D) = R_{(\alpha)} B = B^*. \quad (3.71)$$

Thus, the resulting minimum MSE of d_6 is given by

$$\min. \text{MSE}(d_6) = \min. \text{MSE}(d_5), \quad (3.72)$$

where $\min. \text{MSE}(d_5)$ is given by (3.67). Thus, we see that the family of estimators d_6 is wider than d_5 but its minimum MSE is the same as that of d_5 and is not reduced. It is to be noted that the difference type estimator:

$$d_{6(1)} = \hat{R}_{\alpha(n-r)} + \alpha_1(u^* - 1) + \alpha_2(v^* - 1)$$

is a member of d_6 , but not of d_5 .

Remark 3.8 The following families of estimators of $R_{(\alpha)}$:

$$\begin{aligned} d_{6(1)}^{(1)} &= \hat{R}_{\alpha(n-r)} a(u^*); & d_{6(2)}^{(1)} &= \hat{R}_{\alpha(n-r)} a(v^*); \\ d_{6(3)}^{(1)} &= A(\hat{R}_{\alpha(n-r)}, u^*); & d_{6(4)}^{(1)} &= A(\hat{R}_{\alpha(n-r)}, v^*) \end{aligned}$$

may be identified as members of the family d_6 and their biases and MSEs can be obtained from (3.69) and (3.70), respectively, where $a(u^*)$, $a(v^*)$, $A(\hat{R}_{\alpha(n-r)}, u^*)$ and $A(\hat{R}_{\alpha(n-r)}, v^*)$ are the functions of u^* , v^* , $(\hat{R}_{\alpha(n-r)}, u^*)$ and $(\hat{R}_{\alpha(n-r)}, v^*)$ such that $a(1) = 1$ and $A(R_{(\alpha)}, 1) = R_{(\alpha)}$ which implies $A_1(R_{(\alpha)}, 1) = 1$.

Remark 3.9 If we set $\alpha = 0$ in d_5 , d_6 , $d_{(j)}^{(1)}$, $j = 1, 2, 3, 4$, we get the families of estimators for the population mean $\bar{Y}_0$ as

$$\begin{aligned} d_{6(1)}^{(0)} &= \bar{y}_{0(n-r)} a(u^*, v^*); & d_{6(2)}^{(0)} &= A(\bar{y}_{0(n-r)}, u^*, v^*); & d_{6(3)}^{(0)} &= \bar{y}_{0(n-r)} a(u^*); \\ d_{6(4)}^{(0)} &= \bar{y}_{0(n-r)} a(v^*); & d_{6(5)}^{(0)} &= A(\bar{y}_{0(n-r)}, u^*); & d_{6(6)}^{(0)} &= A(\bar{y}_{0(n-r)}, v^*), \end{aligned}$$

where $a(u^*, v^*)$, $A(\bar{y}_{0(n-r)}, u^*, v^*)$, $a(u^*)$, $a(v^*)$, $A(\bar{y}_{0(n-r)}, u^*)$, $A(\bar{y}_{0(n-r)}, v^*)$ are the functions of (u^*, v^*) , $(\bar{y}_{0(n-r)}, u^*, v^*)$, $(\bar{y}_{0(n-r)}, u^*)$ and $(\bar{y}_{0(n-r)}, v^*)$ such that $a(1, 1) = 1$, $A(D_0) = \bar{Y}_0$ which implies $A_1(D_0) = 1$, $a(1) = 1$, and $A(\bar{Y}_0, 1) = \bar{Y}_0$ which implies that $A_1(\bar{Y}_0, 1) = 1$. Putting $\alpha = 0$ and suitable values of derivatives in (3.65) and (3.70), the biases and MSEs of these estimators can be easily obtained.

The minimum MSEs of the estimators $d_{(j)}^{(1)}$, $j = 1, 2, 3, 4, 5, 6$, are given by

$$\begin{aligned} \min.\text{MSE}(d_{6(1)}^{(0)}) &= \min.\text{MSE}(d_{6(2)}^{(0)}) \\ &= S_0^2 \left[\theta^* - (\theta^* - \theta) \left\{ \rho_{02}^2 + \frac{(\lambda_{003}\rho_{02} - \lambda_{102})^2}{\Delta} \right\} \right], \end{aligned} \quad (3.73)$$

$$\min.\text{MSE}(d_{6(3)}^{(0)}) = \min.\text{MSE}(d_{6(5)}^{(0)}) = S_0^2 [\theta^* - (\theta^* - \theta)(1 - \rho_{02}^2)], \quad (3.74)$$

and

$$\begin{aligned} \min.\text{MSE}(d_{6(4)}^{(0)}) &= \min.\text{MSE}(d_{6(6)}^{(0)}) \\ &= S_0^2 \left[\theta^* - (\theta^* - \theta) \left\{ 1 - \frac{\lambda_{102}^2}{\lambda_{004} - 1} \right\} \right]. \end{aligned} \quad (3.75)$$

Thus, we establish the following theorem:

Theorem 3.9 Up to terms of order n^{-1} ,

- (i) $\text{MSE}(d_{6(1)}^{(0)} \text{ or } d_{6(2)}^{(0)}) \geq S_0^2 \left[\theta^* - (\theta^* - \theta) \left\{ \rho_{02}^2 + \frac{(\lambda_{003}\rho_{02} - \lambda_{102})^2}{\Delta} \right\} \right]$,
with the equality holding if $a_1(1, 1) = A$ and $a_2(1, 1) = B$ for $d_{6(1)}^{(0)}$, and if $A_2(D_0) = \bar{Y}_0 A_{11}$ and $A_3(D_0) = \bar{Y}_0 B_{11}$ for $d_{6(2)}^{(0)}$.
- (ii) $\text{MSE}(d_{6(3)}^{(0)} \text{ or } d_{6(5)}^{(0)}) \geq S_0^2 [\theta^* - (\theta^* - \theta)(1 - \rho_{02}^2)]$,
with the equality holding if $a_1(1) = -K_{02}$ for $d_{6(3)}^{(0)}$, and if $A_2(\bar{Y}_2, 1) = -\bar{Y}_0 K_{02}$ for $d_{6(5)}^{(0)}$.
- (iii) $\text{MSE}(d_{6(4)}^{(0)} \text{ or } d_{6(6)}^{(0)}) \geq S_0^2 \left[\theta^* - (\theta^* - \theta) \left\{ 1 - \frac{\lambda_{102}^2}{\lambda_{004} - 1} \right\} \right]$,
with the equality holding if $a_2(1) = -\lambda_{102} C_0 / (\lambda_{004} - 1)$ for $d_{6(4)}^{(0)}$, and if $A_3(\bar{Y}_0, 1) = -\bar{Y}_0 \lambda_{102} C_0 / (\lambda_{004} - 1)$ for $d_{6(6)}^{(0)}$.

It is to be mentioned that the estimator $d_{6(7)}^{(0)} = \bar{y}_{0(n-r)} + \alpha_3 \bar{y}_2(u^* - 1)$ is a member of $d_{6(5)}^{(0)}$ (or $d_{6(2)}^{(0)}$) but not of $d_{6(3)}^{(0)}$ (or $d_{6(1)}^{(0)}$). It can be shown that to the first degree of approximation

$$\min.\text{MSE}(d_{6(7)}^{(0)}) = \min.\text{MSE}(d_{6(5)}^{(0)}), \quad (3.76)$$

where $\min.\text{MSE}(d_{6(5)}^{(0)})$ is given by (3.74). The estimator $d_{6(7)}^{(0)}$ is due to Singh et al. (2000).

3.3.1 Estimators based on estimated parameters

The estimated optimum values of $a_1(1, 1)$ and $a_2(1, 1)$, based on sample observations, are, respectively, given by

$$\hat{a}_1(1, 1) = \hat{A}_2 \text{ (say)} \quad \text{and} \quad \hat{a}_2(1, 1) = \hat{B}_2 \text{ (say)}. \quad (3.77)$$

Thus, we define a family of estimators (based on estimated optimum values) for $R_{(\alpha)}$:

$$d_5^* = \hat{R}_{\alpha(n-r)} a^*(u^*, v^*, \hat{A}_2, \hat{B}_2), \quad (3.78)$$

where $a^*(\cdot)$ is a function of $(u^*, v^*, \hat{A}_2, \hat{B}_2)$ such that $a^*(Z) = 1$, $a_1^*(Z) = A$, $a_2^*(Z) = B$, $a_3^*(Z) = 0$, and $a_4^*(Z) = 0$. It can be shown that to the first degree of approximation

$$\text{MSE}(d_5^*) = \min.\text{MSE}(d_5),$$

where $\min.\text{MSE}(d_5)$ is given by (3.67).

Furthermore, a wider class of estimators (based on estimated optimum parameters) of $R_{(\alpha)}$ is defined by

$$d_6^* = A^*(\hat{R}_{\alpha(n-r)}, u^*, v^*, \hat{A}_2^*, \hat{B}_2^*), \quad (3.79)$$

where $A^*(\cdot)$ is a function of $(\hat{R}_{\alpha(n-r)}, u^*, v^*, \hat{A}_2^*, \hat{B}_2^*)$ such that $A^*(Q) = R_{(\alpha)}$, $A_1^*(Q) = 1$, $A_2^*(Q) = A^*$, $A_3^*(Q) = B^*$, $A_4^*(Q) = 0$, $A_5^*(Q) = 0$ with $\hat{A}_2^* = \hat{R}_{\alpha(n-r)} \hat{A}_2$ and $\hat{B}_2^* = \hat{R}_{\alpha(n-r)} \hat{B}_2$. It is easy to verify that to the first degree of approximation

$$\text{MSE}(d_6^*) = \min.\text{MSE}(d_5^*) = \min.\text{MSE}(d_6),$$

where $\min.\text{MSE}(d_5)$ is given in (3.67). The estimator

$$d_{6(1)}^* = \hat{R}_{\alpha(n-r)} + \hat{A}_2^*(u^* - 1) + \hat{B}_2^*(v^* - 1)$$

is a particular member of the family d_6^* . The MSE of d_6^* , to terms of order n^{-1} , is same as that of d_5^* .

Similarly, many other families of estimators based on estimated optimum values can be defined with their approximate MSEs formulas.

Below we give consistent estimators of the minimum MSEs of families of estimators.

Theorem 3.10 *Estimators of the minimum MSE of the families of estimators d_5 (or d_6) (or MSE of d_5^* or d_6^*), $d_{6(j)}^{(0)}$, $j = 1, 2$, $d_{6(3)}^{(0)}$ (or $d_{6(5)}^{(0)}$) and $d_{6(4)}^{(0)}$ (or $d_{6(6)}^{(0)}$) are, respectively, given by*

$$\begin{aligned} \min.\widehat{\text{MSE}}(d_j) &= \widehat{\text{MSE}}(d_j^*) \\ &= \min.\widehat{\text{MSE}}(\hat{R}_{\alpha(n-r)}) - (\hat{\theta}^* - \theta) \hat{R}_{\alpha(n-r)}^2 \hat{B}_{\alpha(n-r)}, \end{aligned}$$

$$\begin{aligned}
\min.\widehat{\text{MSE}}(d_{6(1)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{6(2)}^{(0)}) \\
&= s_{0(n-r)}^2 [\hat{\theta}^* - (\hat{\theta}^* - \theta)(\hat{\rho}_{02(n-r)}^2 + \hat{C}_{\alpha(n-r)})], \\
\min.\widehat{\text{MSE}}(d_{6(3)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{6(5)}^{(0)}) \\
&= s_{0(n-r)}^2 [\theta + (\hat{\theta}^* - \theta)(1 - \hat{\rho}_{02(n-r)}^2)],
\end{aligned}$$

and

$$\begin{aligned}
\min.\widehat{\text{MSE}}(d_{6(4)}^{(0)}) &= \min.\widehat{\text{MSE}}(d_{6(6)}^{(0)}) \\
&= s_{0(n-r)}^2 [\theta + (\hat{\theta}^* - \theta)\{1 - \hat{\lambda}_{102(n-r)}^2/(\hat{\lambda}_{004} - 1)\}],
\end{aligned}$$

where $j = 5, 6$ and $\min.\widehat{\text{MSE}}(\hat{R}_{\alpha(n-r)})$ is given in Theorem 3.4.

4 Efficiency comparisons

From (3.5), (3.7), (3.23) and (3.24) we have

$$\text{MSE}(\hat{R}_{\alpha(n-r)}) - \min.\text{MSE}(d_1) = \theta^* R_{(\alpha)}^2 B_{(\alpha)} \geq 0, \quad (4.1)$$

$$\text{MSE}(\hat{R}_{\alpha(n-r)}) - \min.\text{MSE}(d_{1(1)}) = \theta^* R_{(\alpha)}^2 C_2^2 K_{(\alpha)}^2 \geq 0, \quad (4.2)$$

$$\text{MSE}(\hat{R}_{\alpha(n-r)}) - \min.\text{MSE}(d_{1(2)}) = \theta^* R_{(\alpha)}^2 d_{(\alpha)}^2 / (\lambda_{004} - 1) \geq 0, \quad (4.3)$$

$$\min.\text{MSE}(d_{1(1)}) - \min.\text{MSE}(d_1) = \theta^* R_{(\alpha)}^2 \frac{\Delta_2^2}{\Delta} \geq 0, \quad (4.4)$$

$$\begin{aligned}
&\min.\text{MSE}(d_{1(2)}) - \min.\text{MSE}(d_1) \\
&= \theta^* R_{(\alpha)}^2 \frac{\{K_{(\alpha)} C_2 (\lambda_{004} - 1) - d_{(\alpha)} \lambda_{003}\}^2}{(\lambda_{004} - \lambda_{003}^2 - 1)(\lambda_{004} - 1)} \geq 0.
\end{aligned} \quad (4.5)$$

Thus, we have the following inequalities:

$$\min.\text{MSE}(d_1) \leq \min.\text{MSE}(d_{1(1)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}) \quad (4.6)$$

and

$$\min.\text{MSE}(d_1) \leq \min.\text{MSE}(d_{1(2)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}). \quad (4.7)$$

It is well known that

$$\text{Var}(\bar{y}_{0(n-r)}) = \theta^* S_0^2. \quad (4.8)$$

From (3.27), (3.32), (3.33) and (4.8) we have

$$\text{Var}(\bar{y}_{0(n-r)}) - \min.\text{MSE}(d_{1(1)}^{(1)}) = \theta^* S_0^2 \rho_{02}^2 \geq 0, \quad (4.9)$$

$$\text{Var}(\bar{y}_{0(n-r)}) - \min.\text{MSE}(d_{1(2)}^{(1)}) = \theta^* S_0^2 \lambda_{102}^2 / (\lambda_{004} - 1) \geq 0, \quad (4.10)$$

$$\min.\text{MSE}(d_{1(1)}^{(1)}) - \min.\text{MSE}(d_1^{(1)}) = \theta^* S_0^2 (\lambda_{003} \rho_{02} - \lambda_{102})^2 / \Delta \geq 0, \quad (4.11)$$

$$\begin{aligned} & \min.\text{MSE}(d_{1(2)}^{(1)}) - \min.\text{MSE}(d_1^{(1)}) \\ &= \theta^* S_0^2 (\rho_{02}(\lambda_{004} - 1) - \lambda_{102}\lambda_{003})^2 / \Delta \geq 0. \end{aligned} \quad (4.12)$$

Thus, we have the following inequalities:

$$\min.\text{MSE}(d_1^{(1)}) \leq \min.\text{MSE}(d_{1(1)}^{(1)}) \leq \text{Var}(\bar{y}_{0(n-r)}) \quad (4.13)$$

and

$$\min.\text{MSE}(d_1^{(1)}) \leq \min.\text{MSE}(d_{1(2)}^{(1)}) \leq \text{Var}(\bar{y}_{0(n-r)}). \quad (4.14)$$

It follows from (4.13) and (4.14) that the estimator $d_1^{(1)}$ is more efficient than $\bar{y}_{0(n-r)}$, $d_{1(1)}^{(1)}$, $d_{1(2)}^{(1)}$ and Singh et al. (2000) estimator $d_{2(3)}^{(1)}$.

Similarly, the following inequalities can be easily proved:

$$\min.\text{MSE}(d_3) \leq \min.\text{MSE}(d_{4(1)}^{(1)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}) \quad (4.15)$$

and

$$\min.\text{MSE}(d_3) \leq \min.\text{MSE}(d_{4(2)}^{(1)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}). \quad (4.16)$$

Thus, from (4.15) and (4.16) it follows that the suggested family of estimators d_3 (or d_3^* or d_4) is better than the conventional estimators $\hat{R}_{\alpha(n-r)}$, $d_{4(1)}^{(1)}$ (or $d_{4(3)}^{(1)}$) and $d_{4(2)}^{(1)}$ (or $d_{4(4)}^{(1)}$).

Due to (3.58), (3.59), (3.60) and (4.8), the following inequalities hold:

$$\min.\text{MSE}(d_{4(1)}^{(0)}) \leq \min.\text{MSE}(d_{4(3)}^{(0)}) \leq \text{Var}(\bar{y}_{0(n-r)}) \quad (4.17)$$

and

$$\min.\text{MSE}(d_{4(1)}^{(0)}) \leq \min.\text{MSE}(d_{4(4)}^{(0)}) \leq \text{Var}(\bar{y}_{0(n-r)}). \quad (4.18)$$

Thus, the proposed estimator $d_{4(1)}^{(0)}$ (or $d_{4(2)}^{(0)}$) is better than the usual unbiased estimators $\bar{y}_{0(n-r)}$, $d_{4(3)}^{(0)}$ (or $d_{4(5)}^{(0)}$), $d_{4(4)}^{(0)}$ (or $d_{4(6)}^{(0)}$) and Singh et al. (2000) estimator $d_{4(7)}^{(0)}$.

Furthermore, it can be easily proved that

$$\min.\text{MSE}(d_5) \leq \min.\text{MSE}(d_{6(1)}^{(1)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}) \quad (4.19)$$

and

$$\min.\text{MSE}(d_5) \leq \min.\text{MSE}(d_{6(2)}^{(1)}) \leq \text{MSE}(\hat{R}_{\alpha(n-r)}). \quad (4.20)$$

Thus, the proposed family of estimators d_5 (or d_6) is more efficient than $\hat{R}_{\alpha(n-r)}$, $d_{6(1)}^{(1)}$ (or $d_{6(3)}^{(1)}$), and $d_{6(2)}^{(1)}$ (or $d_{6(4)}^{(1)}$).

From (3.73), (3.74), (3.75) and (4.8) it is easy to see that

$$\min.\text{MSE}(d_{6(1)}^{(0)}) \leq \min.\text{MSE}(d_{6(3)}^{(0)}) \leq \text{Var}(\bar{y}_{0(n-r)}) \quad (4.21)$$

and

$$\min.\text{MSE}(d_{6(1)}^{(0)}) \leq \min.\text{MSE}(d_{6(4)}^{(0)}) \leq \text{Var}(\bar{y}_{0(n-r)}). \quad (4.22)$$

It follows from (4.21) and (4.22) that the estimator $d_{6(1)}^{(0)}$ (or $d_{6(2)}^{(1)}$) is more efficient than the estimators $\bar{y}_{0(n-r)}$, $d_{6(3)}^{(0)}$ (or $d_{6(5)}^{(0)}$), $d_{6(4)}^{(0)}$ (or $d_{6(6)}^{(1)}$) and the estimator $d_{6(7)}^{(0)}$ due to Singh et al. (2000). From (3.7), (3.42) and (3.67) we have

$$\min.\text{MSE}(d_3) - \min.\text{MSE}(d_1) = (\theta^* - \theta) R_{(\alpha)}^2 \Delta_2^2 / \Delta \geq 0 \quad (4.23)$$

and

$$\min.\text{MSE}(d_5) - \min.\text{MSE}(d_3) = \theta R_{(\alpha)}^2 B_{(\alpha)} \geq 0. \quad (4.24)$$

It follows from the above inequalities that the proposed family d_1 (or d_1^*) is the best among all the estimators discussed in the present investigation.

5 Conclusion

The paper has suggested families of estimators of the parameter $R_{(\alpha)}$ in the presence of random non-responses together with their biases and mean squared errors. The proposed families include several classes of estimators of the parameters $R_{(\alpha)}$ whose biases and mean squared errors can be obtained easily. Thus, the proposal of families of estimators is justified as they unify several results. It has been shown that the proposed families are better than usual estimators as well as several other families of estimators. It is interesting to note that the families of estimators based on 'estimated optimum values' have the same mean squared errors up to the first degree of approximation as that of optimum estimators in the families which depend upon the unknown population parameters. This paper also provides several families of estimators of the population mean $\bar{Y}_0$ which are better than conventional unbiased estimator and Singh et al. (2000) estimators. Finally, it is found that the proposed family d_1 (or d_1^*) or d_2 (or d_2^*) is the best in the sense that it has least minimum MSE.

Acknowledgements The authors are thankful to the Editor Prof. Enrique Castillo, an Associate Editor, and the two learned referees for their valuable comments; thus, bringing the original manuscript into the present form. The third author, Anwar H. Joarder, acknowledges King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia for providing excellent research facilities.

Appendix. General notations

The following notations have been used throughout the paper:

- N : number of units in the population;
- n : number of units in the sample;
- r : number of sampling units on which information could not be obtained due to random non-responses ($r = 0, 1, 2, \dots, (n - 2)$);

p : the probability of a non-response among the $(n - 2)$ possible values of non-responses;

$$q = (1 - p); \quad \theta = \left(\frac{1}{n} - \frac{1}{N} \right); \quad \theta^* = \left(\frac{1}{nq + 2p} - \frac{1}{N} \right); \quad \delta = (\theta^* - \theta);$$

SRSWOR: simple random sample without replacement;

$(n - r)$: number of remaining units in the sample treated as SRSWOR sample from the population;

$B(\cdot)$: bias of $(\cdot)$;

$MSE(\cdot)$: mean squared error.

For the variates y_0 , y_1 , and y_2 in the population:

$\bar{Y}_i = N^{-1} \sum_{j=1}^N y_{ij}$: the population mean of the i th variate y_i ($i = 0, 1, 2$);

$R_{(\alpha)} = \bar{Y}_0 / \bar{Y}_1^\alpha$ ($\bar{Y}_1 \neq 0$): the population parameter under study;

α : a scalar taking values 0, 1 and -1 ;

C_i : the population coefficient of variation (CV) of the variate y_i ($i = 0, 1, 2$);

$\rho_{il} = S_{il} / (S_i S_l)$, $i \neq l$: the population correlation coefficient between the variates y_i and y_l ($i \neq l = 0, 1, 2$);

$$(N - 1)S_{il} = \sum_{j=1}^N (y_{ij} - \bar{Y}_i)(y_{lj} - \bar{Y}_l); \quad K_{il} = \rho_{il} C_i / C_l \quad (i \neq l = 0, 1, 2);$$

$$A_{(\alpha)} = [C_0^2 + \alpha C_1^2 (\alpha - 2K_{01})]; \quad K_{(\alpha)} = (K_{02} - \alpha K_{12});$$

$$d_{(\alpha)} = (\lambda_{102} - \alpha \lambda_{012} C_1); \quad A_1 = \Delta_1 / (\Delta C_2);$$

$$A^* = R_{(\alpha)} A; \quad B = \Delta_2 / \Delta; \quad B^* = R_{(\alpha)} B;$$

$$A_{11} = \Delta_{11} / (\Delta C_2); \quad B_{11} = \Delta_{22} / \Delta;$$

$$\Delta_1 = [d_{(\alpha)} \lambda_{003} - K_{(\alpha)} (\lambda_{004} - 1) C_2]; \quad \Delta_2 = [K_{(\alpha)} \lambda_{003} C_2 - d_{(\alpha)}];$$

$$\Delta = (\lambda_{004} - \lambda_{003}^2 - 1) > 0; \quad \Delta_{11} = [\lambda_{101} C_0 - K_{02} (\lambda_{004} - 1) C_2];$$

$$\Delta_{22} = [K_{02} \lambda_{003} C_2 - \lambda_{102} C_0]; \quad B_{(\alpha)} = (K_{(\alpha)}^2 C_2^2 + \Delta B^2);$$

$$K = -K_{02}; \quad k^* = -\bar{Y}_0 K_{02} = -\beta_{02} \bar{Y}_2; \quad \beta_{02} = S_{02} / S_2^2;$$

$$\lambda_{m_1 m_2 m_3} = \mu_{m_1 m_2 m_3} / (\mu_{200}^{m_1/2} \mu_{020}^{m_2/2} \mu_{002}^{m_3/2});$$

$$(N - 1) \mu_{m_1 m_2 m_3} = \sum_{j=1}^N (y_{0j} - \bar{Y}_0)^{m_1} (y_{1j} - \bar{Y}_1)^{m_2} (y_{2j} - \bar{Y}_2)^{m_3},$$

$(m_1, m_2, m_3 \text{ non-negative integers}).$

For the variates y_0 , y_1 and y_2 in the sample:

$$\bar{y}_i = n^{-1} \sum_{j=1}^n y_{ij} \quad (i = 0, 1, 2); \quad \hat{R}_{(\alpha)} = \bar{y}_0 / \bar{y}_1^\alpha \quad (\bar{y}_1 \neq 0);$$

$$(n-1)s_i^2 = \sum_{j=1}^n (y_{ij} - \bar{y}_i)^2; \quad (n-r)\bar{y}_{i(n-r)} = \sum_{j=1}^{n-r} y_{ij};$$

$$(n-r-1)s_{i(n-r)}^2 = \sum_{j=1}^{n-r} (y_{ij} - \bar{y}_{i(n-r)})^2: \text{ the conditional unbiased}$$

estimator of S_i^2 ($i = 0, 1, 2$);

$$\hat{R}_{\alpha(n-r)} = \bar{y}_{0(n-r)}/\bar{y}_{1(n-r)}^\alpha \quad (\bar{y}_{1(n-r)} \neq 0); \quad u = \bar{y}_2/\bar{Y}_2; \quad v = s_2^2/S_2^2;$$

$$u_{(n-r)} = \bar{y}_{2(n-r)}/\bar{Y}_2; \quad v_{(n-r)} = s_{2(n-r)}^2/S_2^2;$$

$$u^* = \bar{y}_{2(n-r)}/\bar{y}_2; \quad v^* = s_{2(n-r)}^2/s_2^2;$$

$$\hat{A} = \hat{\Delta}_{1(n-r)}/(\hat{\Delta}_{(n-r)}C_2); \quad \hat{B} = \hat{\Delta}_{2(n-r)}/\hat{\Delta}_{(n-r)};$$

$$\hat{A}^* = \hat{R}_{\alpha(n-r)}\hat{A}; \quad \hat{B}^* = \hat{R}_{\alpha(n-r)}\hat{B};$$

$$\hat{A}_1 = \hat{\Delta}_1/(\hat{\Delta}C_2); \quad \hat{A}_2 = \hat{\Delta}_1^*/(\hat{\Delta}C_2); \quad \hat{B}_1 = \hat{\Delta}_2/\hat{\Delta}; \quad \hat{B}_2 = \hat{\Delta}_2^*/\hat{\Delta};$$

$$\hat{A}_1^* = \hat{R}_{\alpha(n-r)}\hat{A}_1; \quad \hat{B}_1^* = \hat{R}_{\alpha(n-r)}\hat{B}_1;$$

$$\hat{A}_2^* = \hat{R}_{\alpha(n-r)}\hat{A}_2; \quad \hat{B}_2^* = \hat{R}_{\alpha(n-r)}\hat{B}_2;$$

$$\hat{\Delta} = (\hat{\lambda}_{004} - \hat{\lambda}_{003}^2 - 1) > 0; \quad \hat{\Delta}_{(n-r)} = (\hat{\lambda}_{004(n-r)} - \hat{\lambda}_{003(n-r)}^2 - 1);$$

$$\hat{\Delta}_{1(n-r)} = [\hat{d}_{\alpha(n-r)}\hat{\lambda}_{003(n-r)} - \hat{K}_{\alpha(n-r)}(\hat{\lambda}_{004(n-r)} - 1)C_2];$$

$$\hat{\Delta}_{2(n-r)} = [\hat{d}_{\alpha(n-r)}\hat{\lambda}_{003(n-r)}C_2 - \hat{d}_{\alpha(n-r)}];$$

$$\hat{\Delta}_1 = [\hat{d}_{\alpha(n-r)}\hat{\lambda}_{003} - \hat{K}_{(\alpha)}^*(\hat{\lambda}_{004} - 1)C_2];$$

$$\hat{\Delta}_2 = [\hat{K}_{(\alpha)}^*\hat{\lambda}_{003}C_2 - \hat{d}_{\alpha(n-r)}];$$

$$\hat{\Delta}_1^* = [\hat{d}_{\alpha(n-r)}\hat{\lambda}_{003} - \hat{K}_{\alpha(n-r)}^*(\hat{\lambda}_{004} - 1)\hat{C}_2];$$

$$\hat{\Delta}_2^* = [\hat{K}_{\alpha(n-r)}^*\hat{\lambda}_{003}\hat{C}_2 - \hat{d}_{\alpha(n-r)}];$$

$$\hat{K}_{\alpha(n-r)} = [\hat{K}_{02(n-r)} - \alpha\hat{K}_{12(n-r)}];$$

$$\hat{d}_{\alpha(n-r)} = [\hat{\lambda}_{102(n-r)}\hat{C}_{0(n-r)} - \alpha\hat{\lambda}_{012(n-r)}\hat{C}_{1(n-r)}];$$

$$\hat{K}_{(\alpha)}^* = (\hat{K}_{02} - \alpha\hat{K}_{12}); \quad \hat{K}_{\alpha(n-r)}^* = (\hat{K}_{02}^* - \alpha\hat{K}_{12}^*)$$

$$\hat{C}_{0(n-r)} = \sqrt{\hat{\mu}_{200(n-r)}/\bar{y}_{0(n-r)}}; \quad \hat{C}_{1(n-r)} = \sqrt{\hat{\mu}_{020(n-r)}/\bar{y}_{1(n-r)}};$$

$$\hat{C}_{2(n-r)} = \sqrt{\hat{\mu}_{002(n-r)}/\bar{y}_{2(n-r)}};$$

$$\hat{K}_{02(n-r)} = \hat{\rho}_{02(n-r)}\hat{C}_{0(n-r)}/C_2; \quad \hat{K}_{12(n-r)} = \hat{\rho}_{12(n-r)}\hat{C}_{1(n-r)}/C_2;$$

$$\hat{\rho}_{02(n-r)} = \hat{\mu}_{101(n-r)}/\sqrt{\hat{\mu}_{200(n-r)}\hat{\mu}_{002(n-r)}};$$

$$\begin{aligned}
\hat{\rho}_{12(n-r)} &= \hat{\mu}_{011(n-r)} / \sqrt{\hat{\mu}_{020(n-r)} \hat{\mu}_{002(n-r)}}; \\
\hat{K}_{02} &= \hat{\rho}_{02} \hat{C}_{0(n-r)} / C_2; \quad \hat{K}_{12} = \hat{\rho}_{12} \hat{C}_{1(n-r)} / C_2; \\
\hat{\rho}_{02} &= \hat{\mu}_{101(n-r)} / \sqrt{\hat{\mu}_{200(n-r)} \mu_{002}}; \quad \hat{\rho}_{12} = \hat{\mu}_{011(n-r)} / \sqrt{\hat{\mu}_{020(n-r)} \mu_{002}}; \\
\hat{K}_{02}^* &= \{\hat{\mu}_{101(n-r)} \bar{y}_2\} / \{\hat{\mu}_{002} \bar{y}_{0(n-r)}\}; \\
\hat{K}_{12}^* &= \{\hat{\mu}_{011(n-r)} \bar{y}_2\} / \{\hat{\mu}_{002} \bar{y}_{0(n-r)}\}; \quad \hat{C}_2 = \sqrt{\hat{\mu}_{002}} / \bar{y}_2; \\
\hat{\Delta}_{2(n-r)}^* &= \hat{K}_{\alpha(n-r)} \hat{\lambda}_{003(n-r)} \hat{C}_{2(n-r)} - \hat{d}_{\alpha(n-r)}; \\
\hat{K}_{01(n-r)} &= \hat{\rho}_{01(n-r)} \hat{C}_{0(n-r)} / \hat{C}_{1(n-r)}; \\
\hat{\theta}^* &= (1/(n\hat{q} + 2\hat{p}) - 1/N); \quad \hat{\rho}_{01(n-r)} = \hat{\mu}_{110(n-r)} / \sqrt{\hat{\mu}_{200(n-r)} \hat{\mu}_{020(n-r)}};
\end{aligned}$$

and

$$\begin{aligned}
\hat{p} &= \{(n-1+r) - \sqrt{(n-1+r)^2 - 4rn(n-3)/(n-2)}\} / \{2(n-3)\}; \\
\hat{q} &= (1 - \hat{p}); \\
(n-r-1)\hat{\mu}_{m_1 m_2 m_3} &= \sum_{j=1}^{n-r} (y_{0j} - \bar{y}_{0(n-r)})^{m_1} (y_{1j} - \bar{y}_{1(n-r)})^{m_2} (y_{2j} - \bar{y}_{2(n-r)})^{m_3}; \\
\hat{B}_{\alpha(n-r)} &= \hat{K}_{\alpha(n-r)}^2 C_2^2 + \{\hat{K}_{\alpha(n-r)} \hat{\lambda}_{003} \hat{C}_2 - \hat{d}_{\alpha(n-r)}\} / (\hat{\lambda}_{004} - \hat{\lambda}_{003}^2 - 1); \\
\hat{\lambda}_{m_1 m_2 m_3} &= \hat{\mu}_{m_1 m_2 m_3} / \sqrt{\hat{\mu}_{200}^{m_1} \hat{\mu}_{020}^{m_2} \hat{\mu}_{002}^{m_3}} \\
(n-1)\hat{\mu}_{m_1 m_2 m_3} &= \sum_{j=1}^n (y_{0j} - \bar{y}_0)^{m_1} (y_{1j} - \bar{y}_1)^{m_2} (y_{2j} - \bar{y}_2)^{m_3}; \\
\hat{C}_{\alpha(n-r)} &= (\hat{\lambda}_{003} \hat{\rho}_{02(n-r)} - \hat{\lambda}_{102(n-r)})^2 / (\hat{\lambda}_{004} - \hat{\lambda}_{003}^2 - 1), \\
\hat{\lambda}_{m_1 m_2 m_3(n-r)} &= \hat{\mu}_{m_1 m_2 m_3(n-r)} / \sqrt{\hat{\mu}_{200(n-r)}^{m_1} \hat{\mu}_{020(n-r)}^{m_2} \hat{\mu}_{002(n-r)}^{m_3}}; \\
&\quad (m_1, m_2, m_3 \text{ non-negative integers}).
\end{aligned}$$

In addition, we have used the following:

- $H_1(R_{(\alpha)}, 1)$ and $H_2(R_{(\alpha)}, 1)$: the first order partial derivatives of the function $H(\hat{R}_{(\alpha)}, u)$ with respect to (w.r.t.) $\hat{R}_{(\alpha)}$ and u , respectively, about the point $(R_{(\alpha)}, 1)$;
- $H_{ij}(R_{(\alpha)}, 1)$ ($i, j = 1, 2$): the second order partial derivatives of the function $H(\hat{R}_{(\alpha)}, u)$ about the point $(R_{(\alpha)}, 1)$;
- $t_1(1, 1)$ and $t_2(1, 1)$: the first order partial derivatives of the function $t(u, v)$ w.r.t. u and v , respectively, about the point $(1, 1)$;

- $t_{ij}(1, 1)$ ($i, j = 1, 2$): the second order partial derivatives of the function $t(u, v)$ about the point $(1, 1)$;
- $T_1(D)$, $T_2(D)$ and $T_3(D)$: the first order partial derivatives of the function $T(\hat{R}_{(\alpha)}, u, v)$ w.r.t. $\hat{R}_{(\alpha)}$, u and v , respectively, about the point $D = (R_{(\alpha)}, 1, 1)$;
- $T_{ij}(D)$ ($i, j = 1, 2$): the second order partial derivatives of the function $T(\hat{R}_{(\alpha)}, u, v)$ about the point D ;
- $f_1(1, 1)$ and $f_2(1, 1)$: the first order partial derivatives of the function $f(u_{(n-r)}, v_{(n-r)})$ w.r.t. $u_{(n-r)}$ and $v_{(n-r)}$, respectively, about the point $(1, 1)$;
- $f_{ij}(1, 1)$ ($i, j = 1, 2$): the second order partial derivatives of the function $f(u_{(n-r)}, v_{(n-r)})$ about the point $(1, 1)$;
- $(f_1(1), f_{11}(1))$ and $(f_2(1), f_{22}(1))$: the first and second order partial derivatives of the functions $f(u_{(n-r)})$ and $f(v_{(n-r)})$ with respect to $u_{(n-r)}$ and $v_{(n-r)}$, respectively, about the point 'unity';
- $h_1(1)$ and $h_{11}(1)$: the first and the second order partial derivatives of the function $h(u)$ about the point 'unity';
- $F_1(D)$, $F_2(D)$ and $F_3(D)$: the first order partial derivatives of the function $F(\hat{R}_{(\alpha)}, u_{(n-r)}, v_{(n-r)})$ w.r.t. $\hat{R}_{(\alpha)}$, $u_{(n-r)}$ and $v_{(n-r)}$, respectively, about the point D ;
- $F_{ij}(D)$ ($i, j = 1, 2, 3$): the second order partial derivatives of the function $F(\hat{R}_{(\alpha)}, u_{(n-r)}, v_{(n-r)})$ about the point D ;
- $F_1(R_{(\alpha)}, 1)$, $F_2(R_{(\alpha)}, 1)$ and $F_3(R_{(\alpha)}, 1)$: the first order partial derivatives of the functions $(F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}), F(\hat{R}_{\alpha(n-r)}, v_{(n-r)}))$ w.r.t. $\hat{R}_{\alpha(n-r)}$, $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)})$ w.r.t. $u_{(n-r)}$, and $F(\hat{R}_{\alpha(n-r)}, v_{(n-r)})$ w.r.t. $v_{(n-r)}$, respectively, about the point $(R_{(\alpha)}, 1)$;
- $F_{12}(R_{(\alpha)}, 1)$ and $F_{22}(R_{(\alpha)}, 1)$: the second order partial derivatives of the function $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)})$ about the point $(R_{(\alpha)}, 1)$;
- $F_{13}(R_{(\alpha)}, 1)$ and $F_{33}(R_{(\alpha)}, 1)$: the second order partial derivatives of the function $F(\hat{R}_{\alpha(n-r)}, v_{(n-r)})$ about the point $(R_{(\alpha)}, 1)$;
- $f_1^*(Z)$, $f_2^*(Z)$, $f_3^*(Z)$ and $f_4^*(Z)$: the first order partial derivatives of the function $f^*(u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ w.r.t. $u_{(n-r)}$, $v_{(n-r)}$, $\hat{A}$ and $\hat{B}$, respectively, about the point $Z = (1, 1, A, B)$;
- $F_1^*(Q^*)$, $F_2^*(Q^*)$, $F_3^*(Q^*)$, $F_4^*(Q^*)$ and $F_5^*(Q^*)$: the first order partial derivatives of the function $F(\hat{R}_{\alpha(n-r)}, u_{(n-r)}, v_{(n-r)}, \hat{A}, \hat{B})$ w.r.t. $\hat{R}_{\alpha(n-r)}$, $u_{(n-r)}$, $v_{(n-r)}$, $\hat{A}^*$ and $\hat{B}^*$, respectively, about the point $Q^* = (R_{(\alpha)}, 1, 1, A^*, B^*)$;
- $F_1(D_0)$, $F_2(D_0)$ and $F_3(D_0)$: the first order partial derivatives of the function $F(\bar{y}_{0(n-r)}, u_{(n-r)}, v_{(n-r)})$ w.r.t. $\bar{y}_{0(n-r)}$, $u_{(n-r)}$ and $v_{(n-r)}$ about the point $D_0 = (\bar{Y}_0, 1, 1)$;
- $F_{ij}(D_0)$ ($i, j = 1, 2, 3$): the second order partial derivatives of the function $F(\bar{y}_{0(n-r)}, u_{(n-r)}, v_{(n-r)})$ about the point $D_0 = (\bar{Y}_0, 1, 1)$;
- $F_2(\bar{Y}_0, 1)$: the first order partial derivative of the function $F(\bar{y}_{0(n-r)}, u_{(n-r)})$ w.r.t. $u_{(n-r)}$ about the point $(\bar{Y}_0, 1)$;
- $F_{ij}(\bar{Y}_0, 1)$ ($i, j = 1, 2$): the second order partial derivatives of the function $F(\bar{y}_{0(n-r)}, u_{(n-r)})$ about the point $(\bar{Y}_0, 1)$;
- $F_3(\bar{Y}_0, 1)$: the first order partial derivative of the function $F(\bar{y}_{0(n-r)}, v_{(n-r)})$ w.r.t. $v_{(n-r)}$ about the point $(\bar{Y}_0, 1)$;

- $F_{ij}(\bar{Y}_0, 1)$ ($i, j = 1, 3$): the second order partial derivatives of the function $F(\bar{y}_{0(n-r)}, v_{(n-r)})$ about the point $(\bar{Y}_0, 1)$;
- $\phi_1(1, 1)$ and $\phi_2(1, 1)$: the first order partial derivatives of the function $\phi(u, v)$ w.r.t. u and v , respectively, about the point $(1, 1)$;
- $\phi_{ij}(1, 1)$ ($i, j = 1, 2$): the second order partial derivatives of the function $\phi(u, v)$ about the point $(1, 1)$;
- $\Phi_1(D)$, $\Phi_2(D)$ and $\Phi_3(D)$: the first order partial derivatives of the function $\Phi(\hat{R}_{\alpha(n-r)}, u, v)$ w.r.t. $\hat{R}_{\alpha(n-r)}$, u and v , respectively, about the point D ;
- $\Phi_{ij}(D)$ ($i, j = 1, 2, 3$): the second order partial derivatives of the function $\Phi(\hat{R}_{\alpha(n-r)}, u, v)$ about the point D ;
- $\Phi_1(D_0)$, $\Phi_2(D_0)$ and $\Phi_3(D_0)$: the first order partial derivatives of the function $\Phi(\bar{y}_{0(n-r)}, u, v)$ w.r.t. $\bar{y}_{0(n-r)}$, u and v , respectively, about the point D_0 ;
- $\Phi_1(\bar{Y}_0, 1)$: the first order partial derivative of the functions $\Phi(\bar{y}_{0(n-r)}, u)$ and $\Phi(\bar{y}_{0(n-r)}, v)$ w.r.t. $\bar{y}_{0(n-r)}$, about the point $(\bar{Y}_0, 1)$;
- $\Phi_2(\bar{Y}_0, 1)$: the first order partial derivative of the functions $\Phi(\bar{y}_{0(n-r)}, u)$ w.r.t. u about the point $(\bar{Y}_0, 1)$;
- $\Phi_3(\bar{Y}_0, 1)$: the first order partial derivative of the function $\Phi(\bar{y}_{0(n-r)}, v)$ w.r.t. v about the point $(\bar{Y}_0, 1)$;
- $\Phi_1(1)$ and $\Phi_2(1)$: the first order partial derivative of the function $\Phi(u)$ and $\Phi(v)$ w.r.t. u and v , respectively, about the point ‘unity’;
- $\Phi_1(R_{(\alpha)}, 1)$: the first order partial derivative of the functions $\Phi(\hat{R}_{\alpha(n-r)}, u)$ and $\Phi(\hat{R}_{\alpha(n-r)}, v)$ w.r.t. $\hat{R}_{\alpha(n-r)}$ about the point $(R_{(\alpha)}, 1)$;
- $\Phi_1^*(Z)$, $\Phi_2^*(Z)$, $\Phi_3^*(Z)$ and $\Phi_4^*(Z)$: the first order partial derivatives of the function $\Phi^*(u, v, \hat{A}_1^*, \hat{B}_1^*)$ w.r.t. $u, v, \hat{A}_1^*$ and $\hat{B}_1^*$ about the point $(1, 1, A^*, B^*) = Z$;
- $\Phi_1^*(Z^*)$, $\Phi_2^*(Z^*)$, $\Phi_3^*(Z^*)$ and $\Phi_4^*(Z^*)$: the first order partial derivatives of the function $\Phi^*(\hat{R}_{\alpha(n-r)}, u, v, \hat{A}_1^*, \hat{B}_1^*)$ w.r.t. $\hat{R}_{\alpha(n-r)}, u, v, \hat{A}_1^*$ and $\hat{B}_1^*$ about the point $(R_{(\alpha)}, 1, 1, A^*, B^*) = Z^*$;
- $a_1(1, 1)$ and $a_2(1, 1)$: the first order partial derivatives of the function $a(u^*, v^*)$ w.r.t. u^* and v^* , respectively, about the point $(1, 1)$;
- $a_{ij}(1, 1)$ ($i, j = 1, 2$): the first order partial derivatives of the function $a(u^*, v^*)$ about the point $(1, 1)$;
- $A_1(D)$, $A_2(D)$ and $A_3(D)$: the first order partial derivatives of the function $A(\hat{R}_{\alpha(n-r)}, u^*, v^*)$ w.r.t. $\hat{R}_{\alpha(n-r)}$, u^* and v^* , respectively, about the point D ;
- $A_{ij}(D)$ ($i, j = 1, 2, 3$): the second order partial derivatives of the function $A(\hat{R}_{\alpha(n-r)}, u^*, v^*)$ about the point D ;
- $A_1(R_{(\alpha)}, 1)$: the first order partial derivative of the functions $A(\hat{R}_{\alpha(n-r)}, u^*)$ and $A(\hat{R}_{\alpha(n-r)}, v^*)$ with respect to $\hat{R}_{\alpha(n-r)}$ about the point $(R_{(\alpha)}, 1)$;
- $A_1(D_0)$, $A_2(D_0)$ and $A_3(D_0)$: the first order partial derivatives of the function $A(\bar{y}_{0(n-r)}, u^*, v^*)$ w.r.t. $\bar{y}_{0(n-r)}$, u^* and v^* , respectively, about the point D_0 ;
- $A_1(\bar{Y}_0, 1)$: the first order partial derivative of the function $A(\bar{y}_{0(n-r)}, u^*)$ and $A(\bar{y}_{0(n-r)}, v^*)$ w.r.t. $\bar{y}_{0(n-r)}$ about the point $(\bar{Y}_0, 1)$;
- $A_2(\bar{Y}_0, 1)$: the first order partial derivative of the function $A(\bar{y}_{0(n-r)}, u^*)$ w.r.t. u^* about the point $(\bar{Y}_0, 1)$;
- $A_3(\bar{Y}_0, 1)$: the first order partial derivative of the function $A(\bar{y}_{0(n-r)}, v^*)$ w.r.t. v^* about the point $(\bar{Y}_0, 1)$;

- $a_1(1)$: the first order partial derivative of the function $a(u^*)$ w.r.t. u^* about the point 'unity';
- $a_2(1)$: the first order partial derivative of the function $a(v^*)$ w.r.t. v^* about the point 'unity';
- $a_1^*(Z)$, $a_2^*(Z)$, $a_3^*(Z)$ and $a_4^*(Z)$: the first order partial derivatives of the function $a^*(u^*, v^*, \hat{A}_2, \hat{B}_2)$ w.r.t. u^* , v^* , $\hat{A}_2$ and $\hat{B}_2$, respectively, about the point Z ;
- $A_1^*(Q^*)$, $A_2^*(Q^*)$, $A_3^*(Q^*)$, $A_4^*(Q^*)$ and $A_5^*(Q^*)$: the first order partial derivatives of the function $A^*(\hat{R}_{\alpha(n-r)}, u^*, v^*, \hat{A}_2^*, \hat{B}_2^*)$ w.r.t. $\hat{R}_{\alpha(n-r)}$, u^* , v^* , $\hat{A}_2^*$ and $\hat{B}_2^*$, respectively, about the point Q^* .

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