



Characterizations of Correlation Coefficient

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Abstract

The probability density function of correlation coefficient conditional of chi-square variables has been derived. Then the pdf of the correlation coefficient has been easily derived. The conditional moments of correlation coefficient have been derived. The well known moments of correlation coefficient can be easily and quickly derived from the conditional moments. The intermediate steps of the derivation in terms of conditional distributions or moments seem to be worth recording and will be instructive to learners and instructors. A random decomposition of sample correlation coefficient has been discussed. An algorithm has been presented to calculate sample correlation coefficient by using only sample variance.

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Introduction

For a historical account of linear correlation coefficient between two random variables, say X and Y , we refer to Rodgers, J.L. and Nicewander, W.A. (1988) and Stigler (1989). In this paper present a mathematical tutorial of the linear correlation coefficient based on a random sample of size $n \geq 2$.

Let (X_{1j}, X_{2j}) , $j=1, 2, \dots, n$, where $n \geq 2$ be independent samples from a bivariate normal distribution with $a_{ii} = mS_{ii} = mS_i^2 = \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2$, $m = n-1$, $(i=1, 2)$ and

$a_{12} = \sum_{j=1}^n (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)$ which can also be written as $a_{12} = mS_{12} = mRS_1S_2$ where the quantity

$$R = \frac{S_{1,2}}{S_1S_2} = \frac{a_{12}}{(a_{11}a_{22})^{1/2}}$$

is the sample product moment correlation coefficient. The pdf (probability density function) of a_{11}, a_{22} and a_{12} is given by

$$f_1(a_{11}, a_{22}, a_{12}) = \frac{(1-\rho^2)^{-m/2} (\sigma_1\sigma_2)^{-m}}{2^m \sqrt{\pi} \Gamma(m/2) \Gamma((m-1)/2)} (a_{11}a_{22} - a_{12}^2)^{(m-3)/2} \\ (1.1) \times \exp\left(\frac{-a_{11}}{(2-2\rho^2)\sigma_1^2}\right) \exp\left(\frac{-a_{22}}{(2-2\rho^2)\sigma_2^2}\right) \exp\left(\frac{\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right),$$

where $a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, -1 < \rho < 1$, $m \geq 1, \sigma_1 > 0, \sigma_2 > 0$. See e.g. Anderson (2003, 252).

Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ in (1.1) with Jacobian $J((a_{11}, a_{22}, a_{12}) \rightarrow (s_1^2, s_2^2, r)) = m^3 s_1 s_2$, the pdf of S_1^2, S_2^2 and R is given by

$$f_2(s_1^2, s_2^2, r) = \left(\frac{m}{2\sigma_1\sigma_2}\right)^m \frac{(1-\rho^2)^{-m/2}}{\sqrt{\pi} \Gamma(\frac{m}{2}) \Gamma(\frac{m-1}{2})} (1-r^2)^{(m-3)/2} (s_1s_2)^{m-2} \\ \times \exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2} - \frac{ms_2^2}{2(1-\rho^2)\sigma_2^2} + \frac{m\rho r s_1s_2}{(1-\rho^2)\sigma_1\sigma_2}\right). \quad (1.2)$$

By making the transformation $ms_1^2 = \sigma_1^2 u, ms_2^2 = \sigma_2^2 v$, keeping r intact, with Jacobian $J(s_1^2, s_2^2 \rightarrow uv) = (\sigma_1\sigma_2 / m)^2$, we have

$$f_{U,V,R}(u, v, r) = \frac{(1-\rho^2)^{-m/2} (u_1u_2)^{(m-2)/2} (1-r^2)^{(m-3)/2}}{2^m \sqrt{\pi} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-1}{2}\right)} \exp\left(-\frac{u+v}{2(1-\rho^2)} + \frac{\rho r \sqrt{uv}}{1-\rho^2}\right), \\ (1.3)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1$ and $m \geq 1$.

The probability density function of correlation coefficient (R) has been derived by Fisher (1915). Recently Provost (2015) has derived a compact expression of the pdf of R which is better than that by Hotelling (1953). Romero-Padilla (2016) has derived moments of R by a Density Identity Technique due to Joarder (2006). El-Houssainy, Fergany and Edrees (2005) have derived raw and centered moments of R that are easy to approximate for large samples compared to others including Ghosh (1966).

Over more than a century hundreds of research papers have been published of which most have dealt with the application of correlation coefficient in almost all branches of science, engineering, medicine, business etc. Some have dealt with the approximation of moments, confidence intervals and tests. Since the availability of computational facilities, particularly the special functions of mathematics in computer packages, it is important to have exact expressions of moments and other characteristics. In this respect, the contribution by Provost (2015) is indeed significant. But usually the details of the proofs of the derivation of moments are not available. We have developed a systematic derivation of joint probability density functions, conditional probability density functions, conditional moments, marginal moments etc. so that the learning of these things becomes enjoyable in the sense that the level of rigor is milder. We believe this paper will fulfil the leap of inference and benefit learners and instructors in learning important mathematical aspects of correlation coefficient.

The probability density function of correlation coefficient conditional on chi-square variables has been derived in Section 3. The conditional moments of correlation coefficient of positive even and odd orders have been derived in Section 4 and Section 5 respectively. The well-known moments of correlation coefficient are also derived from the conditional moments with much ease. The intermediate steps of the derivation seem to be worth recording and will be instructive to learners and instructors. A random decomposition of sample correlation coefficient has been discussed in Section 6. An algorithm has been presented in Section 7 to calculate sample correlation coefficient by using only sample variance.

2. Mathematical Preliminaries

We will also be using the following mathematical results:

$$a_{\{k\}} = a(a+1)\cdots(a+k-1), \quad a_{\{0\}} = 1. \quad (2.1)$$

$$\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right), \quad x \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots \quad (2.2)$$

Multiplying both sides of (2.2) by $2x$ we have

$$(2x)!\sqrt{\pi} = 2^{2x}x!\Gamma\left(x + \frac{1}{2}\right), \quad x \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots \quad (2.3)$$

Since $(2x+1)! = \Gamma\{2(x+1)\}$, from (2.3), we have

$$(2x+1)!\sqrt{\pi} = 2^{2x}x!\Gamma\left(x + \frac{3}{2}\right), \quad x \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots \quad (2.4)$$

The hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is defined by

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_{\{k\}}(a_2)_{\{k\}} \cdots (a_p)_{\{k\}}}{(b_1)_{\{k\}}(b_2)_{\{k\}} \cdots (b_q)_{\{k\}}} \frac{z^k}{k!}, \quad (2.5)$$

where $a_{\{k\}}$ is defined in (2.1). See for example Gradshteyn and Ryzhik (1994).

The beta function is defined by

$$B(a, b) = \int_0^1 y^{a-1}(1-y)^{b-1} dy, \quad (2.6)$$

which can be evaluated to be

$$B(a, b) = B(b, a) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (2.7)$$

Lemma 2.1 Let $I(a; m) = \int_{-1}^1 r^{k+a}(1-r^2)^{(m-3)/2} dr$ for any non-negative integer k . Then

$$I(a; m) = \frac{1}{2}[(-1)^{k+a} + 1] B\left(\frac{k+a+1}{2}, \frac{m-1}{2}\right). \quad (2.8)$$

Proof. The integral can be written as

$$I(a; m) = \int_{-1}^0 r^{k+a}(1-r^2)^{(m-3)/2} dr + \int_0^1 r^{k+a}(1-r^2)^{(m-3)/2} dr.$$

By substituting $r^2 = y$ so that $r = \pm\sqrt{y}$, we have

$$I(a; m) = \int_{-1}^0 (-\sqrt{y})^{k+a} (1-y)^{(m-3)/2} \frac{-dy}{2\sqrt{y}} + \int_0^1 (\sqrt{y})^{k+a} (1-y)^{(m-3)/2} \frac{dy}{2\sqrt{y}}, \text{ or,}$$

which simplifies to the following:

$$I(a; m) = \frac{1}{2} [(-1)^{k+a} + 1] \int_0^1 y^{(k+a-1)/2} (1-y)^{(m-3)/2} dy.$$

Then (2.8) follows by beta integral from (2.7).

3. The Conditional Distributions of Correlation Coefficient

The following theorem has been derived by Fisher (1915). We provide a brief sketch of the proof with modern notations for the broad spectrum of readers.

Theorem A Let U, V and R have a joint pdf given by (1.3). Then the pdf of R is given by

$$f_R(r) = \frac{2^{m-2} (1-\rho^2)^{m/2} (1-r^2)^{(m-3)/2}}{\pi(m-1)!} \sum_{k=0}^{\infty} \frac{1}{k!} (2\rho r)^k \Gamma^2\left(\frac{m+k}{2}\right), \quad (3.1)$$

where $m \geq 1$ and $-1 < \rho r < 1$.

Proof. By integrating (1.3) over v , we have

$$f_{U,R}(u, r) = \frac{(1-\rho^2)^{-m/2} (u)^{(m-2)/2} (1-r^2)^{(m-3)/2}}{2^m \sqrt{\pi} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-1}{2}\right)} \exp\left(-\frac{u}{2(1-\rho^2)}\right) I(u),$$

$$\text{where } I(u) = \int_0^{\infty} v^{(m-2)/2} \exp\left(-\frac{v}{2(1-\rho^2)}\right) \exp\left(\frac{\rho r \sqrt{uv}}{1-\rho^2}\right) dv.$$

$$\text{Since } I(u) = 2^{m/2} (1-\rho^2)^{m/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho r \sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \Gamma\left(\frac{k+m}{2}\right),$$

$$f_{U,R}(u, r) = \frac{u^{(m-2)/2} (1-r^2)^{(m-3)/2}}{2^{m/2} \sqrt{\pi} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-1}{2}\right)} \exp\left(-\frac{u}{2(1-\rho^2)}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho r \sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \Gamma\left(\frac{k+m}{2}\right). \quad (3.2)$$

By integrating out u , we have

$$f_R(r) = \frac{(1-r^2)^{(m-3)/2}}{2^{m/2} \sqrt{\pi} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-1}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho r \sqrt{2}}{\sqrt{1-\rho^2}} \right)^k \Gamma\left(\frac{k+m}{2}\right) J(k),$$

$$\text{where } J(k) = \int_0^{\infty} u^{(m+k)/2-1} \exp\left(-\frac{u}{2(1-\rho^2)}\right) du.$$

Since $J(k) = \{2(1-\rho^2)\}^{(m+k)/2} \Gamma\left(\frac{m+k}{2}\right)$, by using (2.2) and (2.3), the above pdf simplifies to (3.1).

By using properties of generalized hypergeometric function, Provost (2015) has proved the following expression which is more compact expression than that by Hotelling (1953):

$$f_R(r) = \frac{(m-1)}{2^{m+2} \pi B(m, m+1)} B\left(\frac{m}{2}, \frac{m+1}{2}\right) (1-\rho^2)^{m/2} (1-r^2)^{(m-3)/2} {}_2F_1\left(m, m; m+\frac{1}{2}; \frac{1+\rho r}{2}\right), \quad (3.3)$$

where $m \geq 1$, $-1 < r < 1$, $-1 < \rho < 1$ and ${}_2F_1(a_1, a_2; b_1; z)$ is defined in (2.5).

Theorem 3.1 Let U, V and R have a joint pdf given by (1.3). Then Conditional pdf of R and U given $V = v$ is given by

$$f_{U, R|V}(u, r | v) = \frac{(1-\rho^2)^{-m/2} u^{(m-2)/2} (1-r^2)^{(m-3)/2}}{2^{m/2} \sqrt{\pi} \Gamma\left(\frac{m-1}{2}\right)} \exp\left[-\frac{1}{2(1-\rho^2)}(u + v\rho^2 - \rho r \sqrt{uv})\right], \quad (3.4)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1$ and $m \geq 1$.

Proof. The conditional pdf of $(U, R | V = v)$ is given by

$$f_{R, U|V}(u, r | v) = \frac{f_{U, V, R}(u, v, r)}{f_V(v)}$$

where $f_{U, V, R}(u, v, r)$ is given by (1.3) and $f_V(v)$ is the pdf of V . Since $V \sim \chi_m^2$, the above simplifies to (3.4).

Theorem 3.3 Let then Conditional pdf of R and U given $V = v$ is given by (3.4). Then the Conditional pdf of R given the chi-square variables $U = u$ and $V = v$ is given by

$$f_{R|u,v}(r|u, v) = \frac{(1-\rho^2)^{-m/2}}{B\left(\frac{m-1}{2}, \frac{1}{2}\right)} \exp\left[-\frac{\rho}{2(1-\rho^2)}\{(u+v)\rho - 2r\sqrt{uv}\}\right] (1-r^2)^{(m-3)/2}, \quad (3.5)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1$ and $m \geq 1$.

Proof. The conditional pdf of $(R|U = u, V = v)$ is given by

$$f_{R|u,v}(r|v, u) = \frac{f_{U,R|v}(u, r|v)}{f_U(u)}$$

where $f_{U,R|v}(u, r|v)$ is given by (3.4) and $f_U(u)$ is the pdf of U . Since $U \sim \chi_m^2$, the above then simplifies to (3.5).

Theorem 3.3 Let U and R have a joint pdf given by (3.2). Then Conditional pdf of R given $U = u$ is given by

$$f_{R|u}(r|u) = \frac{(1-r^2)^{(m-3)/2}}{\sqrt{\pi} \Gamma\left(\frac{m-1}{2}\right)} \exp\left(-\frac{u\rho^2}{2(1-\rho^2)}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho r \sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \Gamma\left(\frac{k+m}{2}\right), \quad (3.6)$$

where $u > 0, -1 < r < 1, -1 < \rho < 1$ and $m \geq 1$.

Proof. By definition, we have

$$f_{R|u}(r|u) = \frac{f_{U,R}(u, r)}{f_U(u)}$$

where $f_{U,R}(u, r)$ is given by (3.2) and $f_U(u)$ is the pdf of U . Since $U \sim \chi_m^2$, the above simplifies to (3.3).

Theorem 3.4 Let the conditional pdf of R given $U = u$ is given by (3.6). Then conditional pdf of R given $U = u$ is given by (3.1).

Proof. By definition $f_R(r) = E_U[f_{R|u}(r|U)]$ where $U \sim \chi_m^2$ where $f_{R|u}(r|U)$ is given by (3.6). Then

$$f_R(r) = \int_0^{\infty} f_{R|u}(r|u) f_U(u) du$$

which simplifies to

$$f_R(r) = \frac{2^{m/2-2}(1-r^2)^{(m-3)/2}}{\Gamma(m-1)\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho r \sqrt{2}}{\sqrt{1-\rho^2}} \right)^k \Gamma\left(\frac{k+m}{2}\right) \int_0^{\infty} u^{(m+k)/2-1} \exp\left(-\frac{u}{2(1-\rho^2)}\right) du,$$

where we have used (2.2). The proof then completes by the evaluation of the gamma integral and a little bit of algebraic simplification.

4. Positive Even Moments of Conditioned Correlation Coefficient

Theorem 4.1 Let the Conditional pdf of R given the chi-square variables $U = u$ and $V = v$ is given by (3.5) and a be a positive number. Then the a -th conditional moment of R given the chi-square variables U and V is given by

$$E(R^a | u, v) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{2\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times \sum_{k=0}^{\infty} [(-1)^{k+a} + 1] \frac{1}{k!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2} \right)^k \frac{\Gamma\left(\frac{k+a+1}{2}\right)}{\Gamma\left(\frac{k+a+m}{2}\right)}. \quad (4.1)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1$ and $m \geq 1$.

Proof. By definition, it follows from (3.5) that

$$E(R^a | u, v) = \int_{-1}^1 r^a f_{R|v,u}(r | v, u) dr,$$

which can be expressed as the following:

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{\sqrt{\pi} \Gamma\left(\frac{m-1}{2}\right)} \exp\left(-\frac{u\rho^2}{2(1-\rho^2)}\right) \exp\left(-\frac{v\rho^2}{2(1-\rho^2)}\right) \times \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2} \right)^k I(a; m),$$

where $I(a; m) = \int_{-1}^1 r^{k+a} (1-r^2)^{(m-3)/2} dr$ is defined in Lemma 2.1. The proof is then

completed by algebraic simplifications.

Theorem 4.2 Let the conditional pdf of R given the chi-square variables $U = u$ and $V = v$ be given by (3.5) and a be a positive even number. Then the a -th conditional even moment of R given the chi-square variables U and V is given by

$$E(R^a | v, u) = \frac{\Gamma(m/2)}{(1-\rho^2)^{m/2}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) {}_1F_2\left(\frac{a+1}{2}; \frac{a+m}{2}, \frac{1}{2}; \frac{\rho^2 uv}{4(1-\rho^2)^2}\right), \quad (4.2)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1, m \geq 1$ and ${}_1F_2(a_1; b_1, b_2; z)$ is defined in (2.5).

Proof. By definition,

$$E(R^a | v, u) = \int_{-1}^1 r^a f_{R|u,v}(r | u, v) dr$$

where $f_{R|u,v}(r | u, v)$ is given by (3.5).

Let a be an even number. Then we split the resulting generalized hypergeometric series into two, one with $k = 2j$ and another $k = 2j + 1$ where $j = 0, 1, 2, \dots$ so that the set

$$\{k : k = 0, 1, 2, 3, 4, \dots\} = \{k : k = 0, 2, 4, \dots\} \cup \{k : k = 1, 3, 5, \dots\}$$

is rewritten as

$$\{k : k = 0, 1, 2, 3, 4, \dots\} = \{2j : j = 0, 1, 2, 3, 4, \dots\} \cup \{2j + 1 : j = 0, 1, 2, 3, 4, \dots\}.$$

Then $(-1)^{k+a} + 1 = (-1)^{2j+a} + 1 = 2$ and $(-1)^{k+a} + 1 = (-1)^{2j+1+a} + 1 = 0$

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{2\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times \left[\sum_{j=0}^{\infty} [(-1)^{2j+a} + 1] \frac{1}{(2j)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j} \frac{\Gamma\left(\frac{2j+a+1}{2}\right)}{\Gamma\left(\frac{2j+a+m}{2}\right)} + \sum_{k=0}^{\infty} [(-1)^{2j+1+a} + 1] \frac{1}{(2j+1)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j+1} \frac{\Gamma\left(\frac{2j+1+a+1}{2}\right)}{\Gamma\left(\frac{2j+1+a+m}{2}\right)} \right].$$

so that by (3.5), we have

If a is even, then $2j+1+a$ will be odd, so that $(-1)^{2j+1+a} = -1$, and hence $(-1)^{2j+1+a} + 1 = -1 + 1 = 0$. Then we have

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{2\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times \left[\sum_{j=0}^{\infty} 2 \times \frac{1}{(2j)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j} \frac{\Gamma\left(\frac{2j+a+1}{2}\right)}{\Gamma\left(\frac{2j+a+m}{2}\right)} + 0 \right]$$

which simplifies to the following:

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \sum_{j=0}^{\infty} \frac{1}{(2j)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j} \frac{\Gamma\left(\frac{2j+a+1}{2}\right)}{\Gamma\left(\frac{2j+a+m}{2}\right)}.$$

By using (2.3), (2.4) and (2.5), we have (4.2).

Theorem 4.3 Let U, V and R have a joint pdf given by (1.3). Also let a be a positive even number. Then the a -th conditional moment of R given the chi-square variable $U = u$ is given by

$$E(R^a | u) = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a+m}{2}\right)} \exp\left(-\frac{u\rho^2}{2(1-\rho^2)}\right) {}_2F_2\left(\frac{a+1}{2}, \frac{m}{2}; \frac{a+m}{2}, \frac{1}{2}; \frac{\rho^2 u}{2(1-\rho^2)}\right), \quad (4.3)$$

$u > 0, v > 0, -1 < r < 1, -1 < \rho < 1, m \geq 1$ and ${}_2F_2(a_1, a_2; b_1, b_2; z)$ is defined in (2.5).

Proof. By definition, we have

$$E(R^a | u) = \int_{-1}^1 r^a f_{R|u}(r | u) dr.$$

where $f_{R|u}(r | u)$ is given by (3.6). Then by (2.5) and some algebraic simplifications, we have (4.3).

Theorem 4.4 Let a be positive even number and the a -th conditional moment of R given the chi-square variable $U = u$ is given by (4.3). Then the a -th moment of R is given by

$$E(R^a) = \frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma^2\left(\frac{m}{2}\right)}{\Gamma\left(\frac{a+m}{2}\right)} {}_3F_2\left(\frac{a+1}{2}, \frac{m}{2}, \frac{m}{2}; \frac{a+m}{2}, \frac{1}{2}; \frac{\rho^2}{1+\rho^2}\right), \quad (4.4)$$

where $-1 < r < 1, -1 < \rho < 1, m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5).

Proof. By definition, $E(R^a) = E_U[E(R^a | U)]$ where $U \sim \chi_m^2$. Then (4.4) follows from (4.3) by virtue of (2.5).

The above moment is the equation (20) of Provost (2015) and equation (15) of Romero-Padillo (2016). Provost has proved that the above moment takes much less time than the moment available in Anderson (1984).

The second raw moments of the correlation coefficient R is given by

$$E(R^2) = \frac{(1-\rho^2)^{m/2}}{m} {}_3F_2\left(\frac{m}{2}, \frac{m}{2}, \frac{3}{2}; \frac{m}{2} + 1, \frac{1}{2}; \frac{\rho^2}{1+\rho^2}\right), \quad (4.5)$$

where $-1 < \rho < 1, m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5). The above moment matches with expression (18) of Romero-Padilla (2016). The fourth raw moment of R is given by

$$E(R^4) = \frac{3(1-\rho^2)^{m/2}}{m(m+2)} {}_3F_2\left(\frac{m}{2}, \frac{m}{2}, \frac{5}{2}; \frac{m}{2} + 2, \frac{1}{2}; \frac{\rho^2}{1+\rho^2}\right), \quad (4.6)$$

where $-1 < \rho < 1, m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5).

There are two forms of $E(R^4)$ in expression (20) of Roemro-Paddilla (2016). The first one matches with the above expression in (4.6) but there is an additional 2 in the denominator of the second form.

5. Positive Odd Moments of Conditioned Correlation Coefficient

Theorem 5.1 Let pdf of R given the chi-square variables $U = u$ and $V = v$ be given by (3.5). Also let a be a positive odd number. Then the a -th conditional odd moment of

R given U and V is given by

$$E(R^a | v, u) = \frac{2\Gamma(m/2)\rho}{\sqrt{\pi}(1-\rho^2)^{(m+2)/2}} \frac{\Gamma\left(\frac{a}{2}+1\right)}{\Gamma\left(\frac{m+a+1}{2}\right)} \sqrt{uv} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times {}_2F_1\left(\frac{a}{2}+1, \frac{m+a+1}{2}; \frac{3}{2}; \frac{\rho\sqrt{uv}}{2(1-\rho^2)}\right), \quad (5.1)$$

where $u > 0, v > 0, -1 < r < 1, -1 < \rho < 1, m \geq 1$ and ${}_2F_1(a_1, a_2; b_1; z)$ is defined in (2.5).

Proof. By definition,

$$E(R^a | v, u) = \int_{-1}^1 r^a f_{R|u,v}(r | u, v) dr$$

where $f_{R|u,v}(r | u, v)$ is given by (3.5).

Let a be an odd number. Then we split the resulting generalized hypergeometric series into two, one with $k = 2j$ and another $k = 2j + 1$ where $j = 0, 1, 2, \dots$ so that the set

$\{k : k = 0, 1, 2, 3, 4, \dots\} = \{k : k = 0, 2, 4, \dots\} \cup \{k : k = 1, 3, 5, \dots\}$ is rewritten as

$\{k : k = 0, 1, 2, 3, 4, \dots\} = \{2j : j = 0, 1, 2, 3, 4, \dots\} \cup \{2j + 1 : j = 0, 1, 2, 3, 4, \dots\}.$

Then $(-1)^{k+a} + 1 = (-1)^{2j+a} + 1 = 0$ and $(-1)^{k+a} + 1 = (-1)^{2j+1+a} + 1 = 2$ so that from (3.5), we have

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{2\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times \left[\sum_{j=0}^{\infty} [(-1)^{2j+a} + 1] \frac{1}{(2j)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j} \frac{\Gamma\left(\frac{2j+a+1}{2}\right)}{\Gamma\left(\frac{2j+a+m}{2}\right)} + \sum_{j=0}^{\infty} [(-1)^{2j+1+a} + 1] \frac{1}{(2j+1)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j+1} \frac{\Gamma\left(\frac{2j+1+a+1}{2}\right)}{\Gamma\left(\frac{2j+1+a+m}{2}\right)} \right].$$

The above simplifies to the following:

$$E(R^a | v, u) = \frac{\Gamma(m/2)(1-\rho^2)^{-m/2}}{2\sqrt{\pi}} \exp\left(-\frac{(u+v)\rho^2}{2(1-\rho^2)}\right) \times \left[0 + \sum_{j=0}^{\infty} (2) \frac{1}{(2j+1)!} \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^{2j+1} \frac{\Gamma\left(\frac{2j+1+a+1}{2}\right)}{\Gamma\left(\frac{2j+1+a+m}{2}\right)}\right].$$

Simplifying the factorial notations in the above expression by (2.3) and (2.4), and then using (2.5) with some algebraic simplifications, we have (5.1).

Theorem 5.2 Let the pdf of R given U be given by (3.6) and a be a positive odd number. Then the a -th conditional odd moment of R given the chi-square variable U is given by

$$E(R^a | u) = \frac{2\sqrt{2} \rho\sqrt{u}}{\sqrt{\pi}(1-\rho^2)} \exp\left(-\frac{u\rho^2}{2(1-\rho^2)}\right) \frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+a+1}{2}\right)} \times {}_2F_2\left(\frac{a+2}{2}, \frac{m+1}{2}; \frac{m+a+1}{2}, \frac{3}{2}; \frac{\rho^2 u}{2(1-\rho^2)}\right), \quad (5.2)$$

where $u > 0, -1 < r < 1, -1 < \rho < 1, m \geq 1$ and ${}_2F_2(a_1, a_2; b_1, b_2; z)$ is defined in (2.5).

Proof. By definition

$$E(R^a | u) = \int_{-1}^1 r^a f_{R|u}(r | u) dr$$

where $f_{R|u}(r | u)$ is given by (3.6). By using (2.7), it follows that the above moment is given by (5.2).

Theorem 5.3 Let $E(R^a | u)$ be given by (5.2) where a is a positive odd number and U is a chi-square variable with m degrees of freedom. Then the a -th moment of R is given by

$$E(R^a) = \frac{2\rho(1-\rho^2)^{m/2}}{\Gamma(m/2)\sqrt{\pi}} \frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma^2\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+a+1}{2}\right)} {}_3F_2\left(\frac{a}{2}+1, \frac{m+1}{2}, \frac{m+1}{2}; \frac{m+a+1}{2}, \frac{3}{2}; \rho^2\right), \quad (5.3)$$

where $-1 < \rho < 1$, $m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5).

Proof. By definition $E(R^a) = E_U[E(R^a | U)]$. Since $U \sim \chi_m^2$, it follows from (5.2) that $E(R^a)$ is given by (5.3).

The above moment is the equation (19) of Provost (2015) and equation (16) of Romero-Padillo (2016). Provost has proved that the above moment takes much less time than the moment available in Anderson (1984). The first raw moment of R is given below:

$$E(R) = \frac{2\rho(1-\rho^2)^{m/2}}{(5.4) m \Gamma^2(m/2)} \Gamma^2\left(\frac{m+1}{2}\right) {}_3F_2\left(\frac{m+1}{2}, \frac{m+1}{2}, \frac{3}{2}; \frac{m}{2}+1, \frac{3}{2}; \rho^2\right),$$

where $-1 < \rho < 1$, $m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5). There are 3 forms of $E(R)$ in the expression (17) of Romero-Padilla (2016). The first one matches with the above expression in (5.4). The last two expressions have typographical errors. The third raw moment of R is given by

$$E(R^3) = \frac{6 \Gamma^2\left(\frac{m+1}{2}\right) \rho(1-\rho^2)^{m/2}}{(5.5) m(m+2) \Gamma^2(m/2)} {}_3F_2\left(\frac{m+1}{2}, \frac{m+1}{2}, \frac{5}{2}; \frac{m}{2}+2, \frac{3}{2}; \rho^2\right),$$

where $-1 < \rho < 1$, $m \geq 1$ and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is defined in (2.5). The moment $E(R^3)$ appearing at (19) of Romero-Padilla (2016) have two expressions; the first one matches with the above expression in (5.5). The second one has typographical errors.

6. A Random Decomposition of Sample Correlation Coefficient

We present here a random decomposition of sample correlation coefficient.

Theorem 6.1 Let $Z_1 \square \chi_m^2$ and $Z_2 \square \chi_m^2$ be statistically independent but $U_1 = \frac{mS_1^2}{\sigma_1^2} \sim \chi_m^2$ and $V_2 = \frac{mS_2^2}{\sigma_2^2} \sim \chi_m^2$ are statistically dependent. Then the sample correlation coefficient has the following random decomposition:

$$R = \frac{m(1+\rho)Z_1}{2\sqrt{U_1V_2}} - \frac{m(1-\rho)Z_2}{2\sqrt{U_1V_2}} \quad (6.1)$$

where $-1 < \rho < 1$ and $m \geq 1$.

Proof. The random sample covariance $S_{1,2}$ has the following representations:

$$S_{1,2} = \frac{1}{2}[(1 + \rho)Z_1 - (1 - \rho)Z_2]\sigma_1\sigma_2,$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$. See for example, page 600 of Johnson, Kotz and Balakrishnan (1995). We also have

$$S_1 = \sqrt{U_1} \frac{\sigma_1}{\sqrt{m}} \text{ where } U_1 = \frac{mS_1^2}{\sigma_1^2} \sim \chi_m^2 \text{ and } S_2 = \sqrt{V_2} \frac{\sigma_2}{\sqrt{m}} \text{ where } V_2 = \frac{mS_2^2}{\sigma_2^2} \sim \chi_m^2.$$

Then we have

$$R = \frac{S_{1,2}}{S_1 S_2} = \frac{(1 + \rho)Z_1\sigma_1\sigma_2/2}{\sqrt{U_1} \frac{\sigma_1}{\sqrt{m}} \sqrt{V_1} \frac{\sigma_2}{\sqrt{m}}} - \frac{(1 - \rho)Z_2\sigma_1\sigma_2/2}{\sqrt{U_1} \frac{\sigma_1}{\sqrt{m}} \sqrt{V_1} \frac{\sigma_2}{\sqrt{m}}},$$

which simplifies to (6.1).

7. An Algorithm for Numerical Calculation of Correlation Coefficient

Based on a sample of two variables

x	x_1	x_2	$\dots$	x_n
y	y_1	y_2	$\dots$	y_n

we present the following algorithm to calculate sample correlation coefficient:

STEP 1 Prepare the following table:

$x - a$	$x_1 - a$	$x_2 - a$	$\dots$	$x_n - a$
$y - b$	$y_1 - b$	$y_2 - b$	$\dots$	$y_n - b$
$x + y = w$	w_1	w_2	$\dots$	w_n
$w - c$	$w_1 - c$	$w_2 - c$	$\dots$	$w_n - c$

where $\min(x) = a$, $\min(y) = b$, $x + y = w$, $\min(w) = c$,

If the sample size is more than 10, the above table may be arranged in four columns.

STEP 2 Calculate Variances

We may ignore the third row while calculating variance. Store the transformed data of the first, second and the fourth row in the calculator sequentially, and calculate variance every time. These would be s_1^2 , s_2^2 and s_w^2 respectively as tabulated in the following

table:

Transformed Data	Variance
$x - a$	s_1^2
$y - b$	s_2^2
$w - c$	s_w^2

The reason of transformation of data is to reduce the size of each observation of a variable. This does not change the sample variance as it does not depend on the origin. For example the variables x and $x - a$ have the same sample variance.

STEP 3: Calculate Sample Covariance

$$s_{1,2} = \frac{1}{2}(s_w^2 - s_1^2 - s_2^2).$$

STEP 4: Calculate Correlation Coefficient

$$r = \frac{s_{1,2}}{s_1 s_2}.$$

8. Conclusion

We have tried to present the rigorous mathematical aspects of correlation coefficient in a milder way by introducing conditionality principle both in deriving pdf and moments. We believe this will motivate, benefit and encourage learners and instructors to embark research in the area.

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