

On the characteristic function of the multivariate Pearson type II distribution

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ABSTRACT

The characteristic function of the multivariate Pearson type II distribution has been derived in terms of the generalized hypergeometric function. The marginal and conditional distributions of the multivariate Pearson Type II distribution have been derived. Some properties have been outlined.

1. INTRODUCTION

Lately statisticians have succeeded in developing numerous multivariate distributions shifting the centre of gravity of multivariate distributions from the multivariate normal distribution. Kotz [3] proposed the multivariate Pearson Type II distribution as an extension to the univariate Pearson Type II distribution. He discussed some properties of the distribution including moments and shapes. The distribution can assume many different forms by changing the shape parameter. Johnson [2] graphed surface of the bivariate Pearson type II distribution for the shape parameter $\alpha = 1, 3/2, 2, 3, 4, 6, 10$. He also presented an algorithm for generating the distribution.

The distribution has simple density function, is easy to generate and possesses properties that provide a useful contrast to the multivariate normal distribution. We make a slight modification to the form of the density considered by Kotz [3] so that the marginal and conditional distributions are well identified. In this paper, we find the characteristic function of the multivariate Pearson type II distribution in terms of the well-known generalized hypergeometric function.

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In particular, we consider the following probability density function (p.d.f.)

$$f(x) = C(\alpha, p) |\Omega|^{-1/2} (1 - (x - \theta)' \Omega^{-1} (x - \theta))^{(\alpha-p)/2} \quad (1.1)$$

where x is a $p \times 1$ vector, θ is a $p \times 1$ vector of location parameters, Ω is a $p \times p$ positive definite matrix of scale parameters, $(x - \theta)' \Omega^{-1} (x - \theta) \leq 1$, $\alpha \geq p$ and the normalizing constant $C(\alpha, p)$ is given by

$$C(\alpha, p) = \frac{\Gamma(\alpha/2 + 1)}{\pi^{p/2} \Gamma((\alpha - p)/2 + 1)} \quad (1.2)$$

We will denote the above p.d.f. by $PII(\theta, \Omega; \alpha)$. The p.d.f. of $Z = \Omega^{-1/2} (X - \theta)$ is given by

$$f(z) = C(\alpha, p) (1 - z'z)^{(\alpha-p)/2}, \quad z'z \leq 1 \quad (1.3)$$

which is the p.d.f. of the spherical Pearson type II distribution. When $p = 1$, the p.d.f. of Z_1 defines a univariate Pearson type II distribution with p.d.f.

$$f(z_1) = B^{-1}(1/2, \alpha/2 + 1/2) (1 - z_1^2)^{(\alpha-1)/2}, \quad z_1^2 \leq 1, \alpha > 1 \quad (1.4)$$

(cf. Stuart and Ord [6]), where $B(m, n)$ is the usual beta function.

2. THE MAIN RESULTS

The main results of the paper are presented in this section in the form of two theorems.

THEOREM 2.1 *Let X have the multivariate Pearson type II distribution given by (1.1). Then the characteristic function of X is given by*

$$\phi_X(t) = \exp(it'\theta) {}_0F_1\left(\frac{\alpha+2}{2}; -\frac{||\Omega^{1/2}t||^2}{4}\right) \quad (2.1)$$

where ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is the generalized hypergeometric function defined by

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; y) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{y^k}{k!} \quad (2.2)$$

with $(a)_j = \Gamma(a+j)/\Gamma(a) = a(a+1)\dots(a+j-1)$.

PROOF. The characteristic function of the spherical Pearson type II distribution with p.d.f. in (1.3) is given by

$$\phi_Z = \int_{|z| \leq 1} \exp(it'z) f(z) dz.$$

By an orthogonal transformation $Y = (y_1, y_2, \dots, y_p)' = CZ$

with $y_1 = t'Z / \|t\|$, we have

$$\phi_Z(t) = C(\alpha, p) \int_{|y| \leq 1} \exp(iy_1 \|t\|) (1 - y_1^2)^{(\alpha-p)/2} dy.$$

By integration over $y_2, y_3, \dots, y_p$, we immediately have

$$\begin{aligned} \phi_Z(t) &= C(\alpha, 1) \int_{|y_1| \leq 1} \exp(iy_1 \|t\|) (1 - y_1^2) dy_1 \\ &= 2C(\alpha, 1) \int_{y_1=0}^1 \exp(iy_1 \|t\|) (1 - y_1^2) dy_1. \end{aligned}$$

Now using the following integral (Prudnikov, Brychkov and Marichev [7])

$$\int_0^a (a^2 - x^2)^{c-1} \cos wx \, dx = \frac{\sqrt{\pi}}{2} (2a/w)^{c-1/2} \Gamma(c) J_{c-1/2}(aw)$$

with $a = 1$, $w = \|t\|$ and $c = (\alpha + 1)/2$, we have

$$\phi_Z(t) = 2^{\alpha/2} \Gamma(\alpha/2 + 1) \|t\|^{-\alpha/2} J_{\alpha/2}(\|t\|)$$

where $J_\alpha(t)$ is the Bessel function of the first kind (see e.g. Lebedev [4]). Then by the use of the following relation between the Bessel function of the first kind and the generalized hypergeometric function

$$J_b(w) = \frac{(w/2)^b}{\Gamma(b+1)} {}_0F_1\left(b; -\frac{w^2}{4}\right),$$

we obtain

$$\phi_Z(t) = {}_0F_1\left(\frac{\alpha+2}{2}; -\frac{\|t\|^2}{4}\right)$$

where ${}_0F_1(b; y)$ is the generalized hypergeometric function defined by (2.2).

Finally, the characteristic function of $X = \theta + \Omega^{1/2}Z$ is given by

$$\phi_X(t) = \exp(it'\theta)\phi_Z(\Omega^{1/2}t) = \exp(it'\theta) {}_0F_1\left(\frac{\alpha+2}{2}; -\frac{||\Omega^{1/2}t||^2}{4}\right). \quad \square$$

The marginal and conditional distributions were derived by Johnson [2] by direct integration. We now derive the marginal and conditional distributions of the multivariate Pearson type II distribution with the help of the characteristic function derived in Theorem 2.1. The regression function as well as the conditional covariance matrix is also derived.

THEOREM 2.2 *Let X have the p.d.f. given by (1.1). Partition X , θ , t and Ω as follows:*

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

where $\theta_2, t_2 \in R^q$ ($q < p$) and Ω_{22} is a $q \times q$ positive definite matrix. Then (i) X_2 has a multivariate Pearson type II distribution $PII(\theta_2, \Omega_{22}; \alpha)$ and (ii) X_1 given $X_2 = x_2$ has a multivariate Pearson type II distribution $PII(\theta_{1.2}, \Omega_{11.2}^*; \alpha - q)$ where

$$\theta_{1.2} = \theta_1 + \Omega_{12}\Omega_{22}^{-1}(x_2 - \theta_2), \quad (2.3)$$

$$\text{and} \quad \Omega_{11.2}^* = (1 - (x_2 - \theta_2)'\Omega_{22}^{-1}(x_2 - \theta_2)(\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})). \quad (2.4)$$

PROOF. Setting $t_1 = 0$ in the characteristic function of X in (2.1), we immediately have

$$\phi_{X_2}(t) = \exp(it_2'\theta_2) {}_0F_1\left(\frac{\alpha+2}{2}; -\frac{||\Omega_{22}^{1/2}t_2||^2}{4}\right).$$

This implies that X_2 has a q -variate Pearson type II distribution with the p.d.f.

$$f(x_2) = C(\alpha, q) |\Omega_{22}|^{-1/2} (1 - (x_2 - \theta_2)'\Omega_{22}^{-1}(x_2 - \theta_2))^{(\alpha-q)/2} \quad (2.5)$$

where $(x_2 - \theta_2)'\Omega_{22}^{-1}(x_2 - \theta_2) \leq 1$, $\alpha \geq q$.

It follows from the following two well-known identities:

$$|\Omega| = |\Omega_{22}| |\Omega_{11.2}^*|, \quad \Omega_{11.2}^* = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21} \quad (2.6)$$

$$\text{and } y' \Omega y = y_2' \Omega_{22}^{-1} y_2 + (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)' \Omega_{11.2}^{-1} (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2) \quad (2.7)$$

where $Y = X - \theta$, that

$$\begin{aligned} 1 - y' \Omega y &= 1 - y_2' \Omega_{22}^{-1} y_2 - (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)' \Omega_{11.2}^{-1} (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2) \\ &= (1 - y_2' \Omega_{22}^{-1} y_2) [1 - (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)' V_{11.2}^{-1} (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)] \end{aligned}$$

where $V_{11.2} = (1 - y_2' \Omega_{22}^{-1} y_2) \Omega_{11.2}$. Then the joint density of Y_1 and Y_2 can be written as

$$\begin{aligned} f(y_1, y_2) &= C(\alpha, p) |\Omega_{22}|^{-1/2} |\Omega_{11.2}|^{1/2} (1 - y_2' \Omega_{22}^{-1} y_2)^{(\alpha-p)/2} \\ &\quad \times \left[1 - (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)' V_{11.2}^{-1} (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2) \right]^{(\alpha-p)/2} \end{aligned}$$

so that from (2.5), we immediately have

$$\begin{aligned} f(y_1 | y_2) &= C(\alpha - q, p - q) |V_{11.2}|^{-1/2} \\ &\quad \times \left[1 - (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2)' (1 - y_2' \Omega_{22}^{-1} y_2)^{-1} \right. \\ &\quad \left. V_{11.2}^{-1} (y_1 - \Omega_{12} \Omega_{22}^{-1} y_2) \right]^{(\alpha-p)/2} \end{aligned}$$

From the relation $Y = X - \theta$, we finally have that the conditional distribution of $X_1 | X_2 = x_2$ is a $(p - q)$ -dimensional Pearson type II distribution given by $PII(\theta_{1.2}, \Omega_{11.2}^*; \alpha - q)$ where $\theta_{1.2}$ and $\Omega_{11.2}^*$ are given by (2.3) and (2.4) respectively. $\square$

Following Muirhead [5], it may be verified that $R^2 = Z'Z$ has a $Beta(p, \alpha/2 - p/2)$ distribution. It is also easily checked that $E(X) = \theta$ and $Cov(X) = E(R^2) \Omega / p$ where $E(R^m) = (1/2) B(m + p/2, \alpha/2 - p/2 + 1)$ (see e.g. Cambanis, Huang and Hsu [1]). Thus the regression of X_1 on $X_2 = x_2$ is given by $E(X_1 | X_2 = x_2) = \theta_{1.2}$ where $\theta_{1.2}$ is given by (2.3). The conditional covariance matrix is given by

$$Cov(X_1 | X_2 = x_2) = B(2 + \alpha/2 - q/2, \alpha/2 - p/2 + 1) \Omega_{11.2}^* / (2p).$$

It is seen that, unlike the conditional covariance matrix of the multivariate normal distribution, the conditional covariance matrix of the Pearson type II distribution depends on the conditioned random variable.

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