

Product Moments of Bivariate Wishart Distribution

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ABSTRACT The moments of the multivariate Wishart distribution is known up to the fourth order. But the product moments of the elements, namely, the two sample variances and the correlation coefficient of the bivariate Wishart distribution, are not available in general. In this paper some product moments of an arbitrary order are derived for the three elements of the bivariate Wishart distribution. A theorem containing eight identities of infinite series involving product of two gamma functions has been established to facilitate the derivation. The general nature of the theorem indicates its use in other contexts.

Keywords Bivariate Wishart distribution; Product moments, Correlation coefficient.

1. Introduction

The moments of the multivariate Wishart distribution is known up to the fourth order. Some researchers have derived moments of some special type of functions, namely, sample trace and determinant of Wishart Matrix which have had applications in estimation theory. Only a few special cases of product moments are published in the literature which are mostly considered for applications in correlation analysis. This paper aims at deriving some product moments of the three elements, namely, two sample variances and the correlation coefficient of bivariate Wishart matrix. A theorem containing eight identities of infinite series involving product of two gamma functions has been established to facilitate the derivation. This will lead to generalization of some of the published results in the area.

Let $X_1, X_2, \dots, X_N$ ($N > p$) be a p -dimensional independent normal random vector with mean vector $\bar{X}$ so that the sums of squares and cross product matrix is given by

$$\sum_{j=1}^N (X_j - \bar{X})(X_j - \bar{X})' = A.$$

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The random symmetric positive definite matrix A is said to have a Wishart distribution with parameters p , $m = N - 1 > p$ and $\Sigma(p \times p) > 0$, written as $A \sim W_p(m, \Sigma)$ if its pdf (probability density function) is given by

$$f(A) = \frac{|A|^{(m-p-1)/2} \exp\left(-\frac{1}{2} \text{tr} \Sigma^{-1} A\right)}{2^{mp/2} \pi^{p(p-1)/4} |\Sigma|^{m/2} \prod_{i=1}^p \Gamma\left(\frac{1}{2}(m+1-i)\right)}, \quad A > 0, \quad m > p$$

(see, e.g., Anderson [1], p. 252). The symmetric matrix A for the bivariate case, i.e., for $p = 2$, can be written as $A = (a_{ik})$, $i = 1, 2$; $k = 1, 2$ where

$$a_{ii} = mS_{ii} = mS_i^2 = \sum_{j=1}^N (X_{ij} - \bar{X}_i)^2, \quad m = N - 1, (i = 1, 2),$$

$$a_{12} = \sum_{j=1}^N (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mS_{12} = mRS_1S_2,$$

and the quantity $R = S_{12} / (S_1S_2) = a_{12} / (a_{11}a_{22})^{1/2}$ is the sample product moment correlation coefficient.

We want to derive the moments $E(S_1^{2l_1} S_2^{2l_2} R^l)$ for finite l_1, l_2, l . Some of them may not have closed forms. This paper may help estimate some parametric functions of the elements of $\Sigma = (\sigma_{ik})$, $\sigma_{ii} = \sigma_i^2$, $\sigma_{ik} = \rho \sigma_i \sigma_k$ ($i = 1, 2$; $k = 1, 2$) where $\sigma_1 > 0, \sigma_2 > 0$, and ρ , $(-1 < \rho < 1)$, is the product moment correlation coefficient between X_1 and X_2 .

Fisher [3] derived the distribution of A for $p = 2$ in order to study the distribution of correlation coefficient from a normal sample. Wishart [8] obtained the distribution for arbitrary p as the joint distribution of sample variances and covariances from the multivariate normal population. Because of its important role in multivariate statistical analysis, various authors have derived it from different perspectives (see the references in Gupta and Nagar [5], pp. 87-88).

The first moment of A , $\text{tr}A$, $\det(A)$, A^{-1} and similar beautiful quantities are known (Muirhead [7]). A nice update of moments of the Wishart distribution is given by Gupta and Nagar [5]. But very few product moments can be deduced from the works published so far. Hence we derive some product moments of the bivariate Wishart distribution when the parent variables have the bivariate normal distribution.

2. The General Product Moments of Sample Variances and the Correlation Coefficient

The moments of sample variances S_i^2 ($i = 1, 2$) are well known. The moments of product moment correlation coefficient R are also known though they do not have closed forms.

Ghosh [4] expressed the moments of R in terms of hypergeometric functions. The first moment is the simplest one and is given by

$$E(R) = \frac{2}{m} \left(\frac{\Gamma((m+1)/2)}{\Gamma(m/2)} \right)^2 \rho {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{m+2}{2}; \rho^2 \right), \quad -1 < \rho < 1, m > 1.$$

There are a number of representations of the moments of the correlation coefficient in Johnson *et al.* ([6], p.553). Understandably the derivation of the joint moments of $S_1^{2l_1}, S_2^{2l_2}$ and R^l will be formidably difficult. By differentiating the moment generating function of $A \sim W_p(m, \Sigma)$, $m > p$, de Waal and Nel [2] derived the following results:

$$(a) E(A^2) = m \left((m+1)\Sigma + (tr \Sigma) I_p \right) \Sigma,$$

$$(b) E(A^3) = m \left((m^2 + 3m + 4)\Sigma^2 + 2(m+1)(tr \Sigma)\Sigma + (m+1)(tr \Sigma^2)I_p + (tr \Sigma)^2 I_p \right) \Sigma.$$

Some product moments, e.g., $E(S_1 S_2^3 R)$, $E(S_1^3 S_2 R)$ and $E(S_1^2 S_2^2 R^2)$, can be deduced from the above identity in (a).

Theorem 2.1 For finite l_1 , l_2 , and l , the product moments $E(S_1^{2l_1} S_2^{2l_2} R^l)$ denoted by $\mu'(l_1, l_2, l; \rho)$ are given by

$$\begin{aligned} \mu'(l_1, l_2, l; \rho) &= \frac{2^{l_1+l_2}}{m^{l_1+l_2} L(m, \rho)} (1-\rho^2)^{l_1+l_2} \sigma_1^{2l_1} \sigma_2^{2l_2} \\ &\times \sum_{k=0}^{\infty} \rho^k \frac{2^k}{k!} \Gamma\left(\frac{k+m}{2} + l_1\right) \Gamma\left(\frac{k+m}{2} + l_2\right) \frac{\Gamma\left(\frac{k+1+l}{2}\right)}{\Gamma\left(\frac{k+m+l}{2}\right)}, \end{aligned} \quad (2.1)$$

where $L(m, \rho) = \sqrt{\pi} \Gamma(m/2) (1-\rho^2)^{-m/2}$, $m > 2$, $\sigma_1 > 0$, $\sigma_2 > 0$, and $-1 < \rho < 1$.

Proof. The pdf of the elements of A can be written as

$$\begin{aligned} f_1(a_{11}, a_{22}, a_{12}) &= \frac{(1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{2^m \sqrt{\pi} \Gamma(m/2) \Gamma((m-1)/2)} (a_{11} a_{22} - a_{12}^2)^{(m-3)/2} \\ &\times \exp\left(-\frac{1}{2} \frac{a_{11}}{(1-\rho^2)\sigma_1^2}\right) \exp\left(-\frac{1}{2} \frac{a_{22}}{(1-\rho^2)\sigma_2^2}\right) \exp\left(\frac{\rho a_{12}}{(1-\rho^2)\sigma_1 \sigma_2}\right) \end{aligned}$$

where $a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, -1 < \rho < 1, m > 2, \sigma_1 > 0, \sigma_2 > 0$. Under the transformation $a_{11} = m s_1^2, a_{22} = m s_2^2, a_{12} = m r s_1 s_2$ with Jacobian $m^3 s_1 s_2$, the joint pdf of S_1^2, S_2^2 and R is given by

$$f_2(s_1^2, s_2^2, r) = \frac{m^m (1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{4\pi \Gamma(m-1)} s_1^{m-2} \exp\left(-\frac{1}{2} \frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right) \times s_2^{m-2} \exp\left(-\frac{1}{2} \frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right) (1-r^2)^{(m-3)/2} \exp\left(\frac{m\rho rs_1 s_2}{(1-\rho^2)\sigma_1 \sigma_2}\right) \quad (2.2)$$

where we have also used the duplication formula of gamma function

$$\Gamma(z) = \frac{2^{z-1}}{\sqrt{\pi}} \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{z}{2}\right)$$

with $z = m-1$. By expanding the last term, we have

$$f_3(s_1^2, s_2^2, r) = \frac{m^m (1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{4\pi \Gamma(m-1)} \exp\left(-\frac{m}{2(1-\rho^2)} \left(\frac{s_1^2}{\sigma_1^2} + \frac{s_2^2}{\sigma_2^2}\right)\right) \times \sum_{k=0}^{\infty} \frac{(m\rho)^k (s_1 s_2)^{m-2+k}}{(1-\rho^2)^k (\sigma_1 \sigma_2)^k k!} r^k (1-r^2)^{(m-3)/2}.$$

Then the product moments $\mu'(l_1, l_2, l; \rho)$ are given by

$$\mu'(l_1, l_2, l; \rho) = \frac{m^m (1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{4\pi \Gamma(m-1)} \times \sum_{k=0}^{\infty} \frac{(m\rho)^k}{(1-\rho^2)^k (\sigma_1 \sigma_2)^k k!} \int_0^{\infty} s_1^{m+k+2l_1-2} \exp\left(-\frac{1}{2} \frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right) ds_1^2 \times \int_0^{\infty} s_2^{m+k+2l_2-2} \exp\left(-\frac{1}{2} \frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right) ds_2^2 \int_{-1}^1 r^{k+l} (1-r^2)^{(m-3)/2} dr.$$

That is

$$\mu'(l_1, l_2, l; \rho) = \frac{m^m (1-\rho^2)^{-m/2} (\sigma_1 \sigma_2)^{-m}}{4\pi \Gamma(m-1)} \times \sum_{k=0}^{\infty} \frac{(m\rho)^k}{(1-\rho^2)^k (\sigma_1 \sigma_2)^k k!} \int_0^{\infty} (s_1^2)^{(m+k)/2+l_1-1} \exp\left(-\frac{1}{2} \frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right) ds_1^2 \times \int_0^{\infty} (s_2^2)^{(m+k)/2+l_2-1} \exp\left(-\frac{1}{2} \frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right) ds_2^2 \int_0^1 u^{(k+l-1)/2} (1-u)^{(m-3)/2} du$$

which can further be evaluated as

$$\begin{aligned} \mu'(l_1, l_2, l; \rho) &= \frac{2^{l_1+l_2+m}}{4\pi m^{l_1+l_2}} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(m-1)} (1-\rho^2)^{l_1+l_2+m/2} \sigma_1^{2l_1} \sigma_2^{2l_2} \\ &\times \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2}+l_1\right) \Gamma\left(\frac{k+m}{2}+l_2\right) \frac{\Gamma\left(\frac{k+1+l}{2}\right)}{\Gamma\left(\frac{k+m+l}{2}\right)}. \end{aligned}$$

The pdf of the sample correlation coefficient originally derived by Fisher [3] has the following representation.

Theorem 2.2 The probability density function of the correlation coefficient R is given by

$$h(r) = \frac{2^{m-2} \Gamma^2(m/2) (1-\rho^2)^{m/2}}{\pi \Gamma(m-1)} (1-r^2)^{(m-3)/2} E(e^{\rho r \sqrt{U_1 U_2}}), \quad -1 < r < 1$$

where $m > 2, -1 < \rho < 1$ and the expectation is over the variables $U_i \sim \chi_m^2$ ($i = 1, 2$).

Proof. By making the transformation

$$\frac{ms_1^2}{\sigma_1^2(1-\rho^2)} = u_1, \quad \frac{ms_2^2}{\sigma_2^2(1-\rho^2)} = u_2$$

with Jacobian $J(s_1^2, s_2^2 \rightarrow u_1, u_2) = (\sigma_1 \sigma_2 / m)^2 (1-\rho^2)^2$ in (2.2), we have

$$f_4(u_1, u_2, r) = \frac{(1-\rho^2)^{m/2}}{4\pi \Gamma(m-1)} (1-r^2)^{(m-3)/2} e^{\rho r \sqrt{u_1 u_2}} (u_1 u_2)^{(m/2)-1} e^{-(u_1+u_2)/2}.$$

Then the theorem is obvious by the integration over u_1 and u_2 as the following:

$$h(r) = \frac{(1-\rho^2)^{m/2}}{4\pi \Gamma(m-1)} (1-r^2)^{(m-3)/2} \int_0^\infty \int_0^\infty e^{\rho r \sqrt{u_1 u_2}} (u_1 u_2)^{(m/2)-1} e^{-(u_1+u_2)/2} du_1 du_2.$$

Note that by expanding the term $e^{\rho r \sqrt{u_1 u_2}}$ and completing the integration over u_1 and u_2 we have the following well known probability density function of R :

$$h(r) = \frac{2^{m-2} (1-\rho^2)^{m/2}}{\pi \Gamma(m-1)} (1-r^2)^{(m-3)/2} \sum_{k=0}^{\infty} \frac{(2\rho r)^k}{k!} \Gamma^2\left(\frac{m+k}{2}\right), \quad -1 < r < 1$$

where $m > 2, -1 < \rho < 1$ (Muirhead [7], p. 154).

3. Some Special Cases of Product Moments $\mu(l_1, l_2, l; \rho)$

To evaluate the product moments given by Theorem 2.1 for some special cases we need the following theorem. In what follows, for any nonnegative integer k , we define:

$$c^{\{k\}} = c(c-1)\cdots(c-k+1), \quad c^{\{0\}} = 1.$$

Theorem 3.1 Let $b_{k,m} = \frac{2^k}{k!} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{k+m}{2}\right)$, $m > 2$, $k > 0$. Then for $L(m, \rho)$, $m > 2$, $-1 < \rho < 1$ defined in Theorem 2.1, we have

$$(i) \quad \sum_{k=0}^{\infty} \rho^k b_{k,m} = L(m, \rho),$$

$$(ii) \quad \sum_{k=0}^{\infty} k \rho^k b_{k,m} = m \rho^2 (1 - \rho^2)^{-1} L(m, \rho) = w_1(m, \rho) L(m, \rho),$$

$$(iii) \quad \sum_{k=0}^{\infty} k^{\{2\}} \rho^k b_{k,m} = (m(m+1)\rho^4 + m\rho^2)(1 - \rho^2)^{-2} L(m, \rho) = w^{\{2\}}(m, \rho) L(m, \rho),$$

$$(iv) \quad \sum_{k=0}^{\infty} k^{\{3\}} \rho^k b_{k,m} = w^{\{3\}}(m, \rho) L(m, \rho) \quad \text{where}$$

$$\begin{aligned} w^{\{3\}}(m, \rho) &= ((m^3 + 3m^2 + 2m)\rho^6 + (3m^2 + 6m)\rho^4)(1 - \rho^2)^{-3} \\ &= (m(m+1)(m+2)\rho^6 + 3m(m+2)\rho^4)(1 - \rho^2)^{-3}, \end{aligned}$$

$$(v) \quad \sum_{k=0}^{\infty} k^{\{4\}} \rho^k b_{k,m} = w^{\{4\}}(m, \rho) L(m, \rho) \quad \text{where}$$

$$\begin{aligned} w^{\{4\}}(m, \rho) &= m[(m^3 + 6m^2 + 11m + 6)\rho^8 + (6m^2 + 30m + 36)\rho^6 + (3m + 6)\rho^4](1 - \rho^2)^{-4} \\ &= [m(m+1)(m+2)(m+3)\rho^8 + 6m(m+2)(m+3)\rho^6 + 3m(m+2)\rho^4](1 - \rho^2)^{-4}, \end{aligned}$$

$$(vi) \quad \sum_{k=0}^{\infty} k^2 \rho^k b_{k,m} = (m^2 \rho^4 + 2m \rho^2)(1 - \rho^2)^{-2} L(m, \rho) = w_2(m, \rho) L(m, \rho),$$

$$(vii) \quad \sum_{k=0}^{\infty} k^3 \rho^k b_{k,m} = (4m \rho^2 + (6m^2 + 4m)\rho^4 + m^3 \rho^6)(1 - \rho^2)^{-3} L(m, \rho) = w_3(m, \rho) L(m, \rho),$$

$$\begin{aligned} (viii) \quad \sum_{k=0}^{\infty} k^4 \rho^k b_{k,m} &= w_4(m, \rho) L(m, \rho) \quad \text{where} \quad w_4(m, \rho) = [(m^4 + 18m^2 + 12m)\rho^8 + (12m^3 \\ &\quad - 20m^2 + 8m)\rho^6 + (46m^2 - 4m)\rho^4 + 32m\rho^2](1 - \rho^2)^{-4}. \end{aligned}$$

Proof. Since

$$\mu'(0,0,0;\rho) = \frac{1}{L(m,\rho)} \sum_{k=0}^{\infty} \rho^k b_{k,m},$$

the proof of (i) is obvious by virtue of $\mu'(0,0,0) = 1$ where $\mu'(l_1, l_2, l; \rho)$ is defined in (2.1). The identity in (i) can be rewritten as

$$\sum_{k=0}^{\infty} \rho^k b_{k,m} = L(m,0)(1-\rho^2)^{-m/2}. \quad (3.1)$$

Differentiating both sides of the identity in Theorem 3.1 (i) with respect to ρ we have

$$\sum_{k=0}^{\infty} k \rho^{k-1} b_{k,m} = mL(m,0)\rho(1-\rho^2)^{-m/2-1} \quad (3.2)$$

which yields the identity in (ii) above. Differentiating the identity in (3.2) again we have

$$\begin{aligned} \sum_{k=0}^{\infty} k(k-1)\rho^{k-2} b_{k,m} &= mL(m,0) \left[\rho \left\{ (-m/2-1)(1-\rho^2)^{-m/2-2} (-2\rho) \right\} + (1-\rho^2)^{-m/2-1} \right] \\ &= mL(m,0) [1 + (m+1)\rho^2] (1-\rho^2)^{-m/2-2}. \end{aligned} \quad (3.3)$$

which yields the identity in (iii) above. Differentiating the identity in (3.3) we have

$$\begin{aligned} \sum_{k=0}^{\infty} k(k-1)(k-2)\rho^{k-3} b_{k,m} &= mL(m,0) \left[(1 + (m+1)\rho^2) \left\{ (-m/2-2)(1-\rho^2)^{-m/2-3} (-2\rho) \right\} + \{(m+1)2\rho\} (1-\rho^2)^{-m/2-2} \right] \\ &= mL(m,0) [(m^2 + 3m + 2)\rho^3 + (3m + 6)\rho] (1-\rho^2)^{-m/2-3} \end{aligned} \quad (3.4)$$

which yields the identity in (iv). Differentiating the identity in (3.4) we have

$$\begin{aligned} \sum_{k=0}^{\infty} k(k-1)(k-2)(k-3)\rho^{k-4} b_{k,m} &= mL(m,0) \left[((m^2 + 3m + 2)\rho^3 + (3m + 6)\rho) \left\{ (-m/2-3)(1-\rho^2)^{-m/2-4} (-2\rho) \right\} \right. \\ &\quad \left. + \{(m^2 + 3m + 2)3\rho^2 + (3m + 6)\} (1-\rho^2)^{-m/2-3} \right] \\ &= mL(m,0) [(m^3 + 6m^2 + 11m + 6)\rho^4 + (6m^2 + 30m + 36)\rho^2 + (3m + 6)] (1-\rho^2)^{-m/2-4} \end{aligned}$$

which yields the identity in (v). The identity in (vi) can be proved as follows:

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 \rho^k b_{k,m} &= \sum_{k=0}^{\infty} (k^{\{2\}} + k) \rho^k b_{k,m} \\
&= (w^{\{2\}}(m, \rho) + w_1(m, \rho)) L(m, \rho) \\
&= \left[w^{\{2\}}(m, \rho) (1 - \rho^2)^2 + w_1(m, \rho) (1 - \rho^2)^2 \right] L(m, \rho) (1 - \rho^2)^{-2} \\
&= \left[((m^2 + m) \rho^4 + m \rho^2) + m \rho^2 (1 - \rho^2) \right] L(m, \rho) (1 - \rho^2)^{-2}
\end{aligned}$$

The identity in (vii) can be proved as follows:

$$\begin{aligned}
\sum_{k=0}^{\infty} k^3 \rho^k b_{k,m} &= \sum_{k=0}^{\infty} (k^{\{3\}} + 3k^{\{2\}} + k) \rho^k b_{k,m} \\
&= (w^{\{3\}}(m, \rho) + 3w^{\{2\}}(m, \rho) + w_1(m, \rho)) L(m, \rho) \\
&= \left[w^{\{3\}}(m, \rho) (1 - \rho^2)^3 + 3w^{\{2\}}(m, \rho) (1 - \rho^2)^3 + w_1(m, \rho) (1 - \rho^2)^3 \right] L(m, \rho) (1 - \rho^2)^{-2} \\
&= \left[((m^3 + 3m^2 + 2m) \rho^6 + (3m^2 + 6m) \rho^4) + 3((m^2 + m) \rho^4 + m \rho^2) (1 - \rho^2) \right. \\
&\quad \left. + m \rho^2 (1 - \rho^2)^2 \right] L(m, \rho) (1 - \rho^2)^{-3}.
\end{aligned}$$

The identity in (viii) can be proved as follows:

$$\begin{aligned}
\sum_{k=0}^{\infty} k^4 \rho^k b_{k,m} &= \sum_{k=0}^{\infty} (k^{\{4\}} + 6k^{\{3\}} + 7k^{\{2\}} + k) \rho^k b_{k,m} \\
&= (w^{\{4\}}(m, \rho) + 6w^{\{3\}}(m, \rho) + 7w^{\{2\}}(m, \rho) + w_1(m, \rho)) L(m, \rho) \\
&= \left[w^{\{4\}}(m, \rho) (1 - \rho^2)^4 + 6w^{\{3\}}(m, \rho) (1 - \rho^2)^4 + 7w^{\{2\}}(m, \rho) (1 - \rho^2)^4 \right. \\
&\quad \left. + w_1(m, \rho) (1 - \rho^2)^4 \right] L(m, \rho) (1 - \rho^2)^{-4} \\
&= \left[((m^4 + 6m^3 + 11m^2 + 6m) \rho^8 + (6m^3 + 30m^2 + 36m) \rho^6 + (3m^2 + 6m) \rho^4) \right. \\
&\quad \left. + 6(m^3 + 3m^2 + 2m) \rho^6 + (3m^2 + 6m) \rho^4 \right] (1 - \rho^2) \\
&\quad + 7((m^2 + m) \rho^4 + m \rho^2) (1 - \rho^2)^2 + m \rho^2 (1 - \rho^2)^3 \Big] L(m, \rho) (1 - \rho^2)^{-4} \\
&= w_4(m, \rho) L(m, \rho)
\end{aligned}$$

where $w^{\{i\}}(m, \rho)$ and $w_i(m, \rho)$, ($i = 1, 2, 3, 4$) are defined in the theorem.

By Direct application of Theorem 3.1 to Theorem 2.1, special cases of moments $\mu'(l_1, l_2, l; \rho)$ of Wishart distribution having probability density function given by (2.2) can be calculated. Some are tabulated below where in general $m > 2, \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1$.

Case I: (One order is negative)

$$\mu'(-2, 0, 0; \rho) = \frac{m^2 \sigma_1^{-4}}{(m-2)(m-4)}, \quad m > 4,$$

$$\begin{aligned} \mu'(-2, 1, 0; \rho) &= \frac{m}{(m-2)(m-4)} (-4\rho^2 + m) \sigma_1^{-4} \sigma_2^2 \\ &= m \left(\frac{\rho^2}{(m-2)\rho^2} + \frac{m}{(m-2)(m-4)} \right) \sigma_1^{-4} \sigma_2^2, \quad m > 4, \end{aligned}$$

$$\mu'(-2, 1, 2; \rho) = m \frac{1 + (m-5)\rho^2}{(m-2)(m-4)} \sigma_1^{-4} \sigma_2^2, \quad m > 4,$$

$$\begin{aligned} \mu'(-2, 2, 0; \rho) &= \left(24\rho^4 - 8(m+2)\rho^2 + m(m+2) \right) \frac{(\sigma_2 / \sigma_1)^4}{(m-2)(m-4)} \\ &= \left(\frac{24(1-\rho^2)^2}{(m-2)(m-4)} + \frac{8(1-\rho^2)}{m-2} + 1 \right) (\sigma_2 / \sigma_1)^4, \quad m > 4, \end{aligned}$$

$$\begin{aligned} \mu'(-2, 2, 2; \rho) &= \left(6(5-m)\rho^4 + (m^2 - m - 24)\rho^2 + (m+2) \right) \frac{(\sigma_2 / \sigma_1)^4}{(m-2)(m-4)} \\ &= \left(\frac{6(1-\rho^2)^2}{(m-2)(m-4)} + \frac{(1+6\rho^2)(1-\rho^2)}{m-2} + \rho^2 \right) (\sigma_2 / \sigma_1)^4, \quad m > 4, \end{aligned}$$

$$\begin{aligned} \mu'(-2, 2, 4; \rho) &= \left((m^2 - 12m + 35)\rho^4 + 6(m-5)\rho^2 + 3 \right) \frac{(\sigma_2 / \sigma_1)^4}{(m-2)(m-4)} \\ &= \left(\frac{3(1-\rho^2)^2}{(m-2)(m-4)} + \frac{6\rho^2(1-\rho^2)}{m-2} + \rho^4 \right) (\sigma_2 / \sigma_1)^4, \quad m > 4, \end{aligned}$$

$$\mu'(-1, 0, 0; \rho) = \frac{m \sigma_1^{-2}}{(m-2)}, \quad m > 2,$$

$$\mu'(-1, 1, 0; \rho) = \frac{m-2\rho^2}{m-2} \left(\frac{\sigma_2}{\sigma_1} \right)^2 = \left(\frac{2(1-\rho^2)}{m-2} + 1 \right) \left(\frac{\sigma_2}{\sigma_1} \right)^2, \quad m > 2,$$

$$\mu'(-1, 1, 2; \rho) = \frac{(m-3)\rho^2 + 1}{m-2} \left(\frac{\sigma_2}{\sigma_1} \right)^2 = \left(\rho^2 + \frac{1-\rho^2}{m-2} \right) \left(\frac{\sigma_2}{\sigma_1} \right)^2, \quad m > 2,$$

$$\mu'(-1, 2, 0; \rho) = \left(8\rho^4 - 2(m+3)\rho^2 + m(m+2) \right) \frac{\sigma_1^{-2} \sigma_2^4}{m(m-2)}, \quad m > 2,$$

$$\mu'(-1, 2, 2; \rho) = \left((-2m+6)\rho^4 + (m^2 + m - 14)\rho^2 + (m+2) \right) \frac{\sigma_1^{-2} \sigma_2^4}{m(m-2)}, \quad m > 2,$$

$$\mu'(0, -2, 0; \rho) = \frac{m^2}{(m-2)(m-4)} \sigma_2^{-4}, \quad m > 4,$$

$$\mu'(0, -1, 0; \rho) = \frac{m\sigma_2^{-2}}{m-2}, \quad m > 2,$$

$$\mu'(1, -2, 2; \rho) = \frac{m((m-5)\rho^2 + 1)}{(m-2)(m-4)} \sigma_1^2 \sigma_2^{-4}, \quad m > 4,$$

$$\mu'(2, -1, 0; \rho) = (8\rho^4 - 4(m+2)\rho^2 + m(m+2)) \frac{1}{m(m-2)} \sigma_1^4 \sigma_2^{-2}, \quad m > 2,$$

$$\mu'(2, -1, 2; \rho) = (-4(m-3)\rho^4 + (m^2 + m - 14)\rho^2 + (m+2)) \frac{1}{m(m-2)} \sigma_1^4 \sigma_2^{-2}, \quad m > 2.$$

Case II: (Each order is nonnegative)

$$\mu'(0, 1, 0; \rho) = \sigma_2^2,$$

$$\mu'(0, 1, 2; \rho) = ((m-1)\rho^2 + 1) \frac{\sigma_2^2}{m},$$

$$\mu'(0, 2, 2; \rho) = (-2(m-1)\rho^4 + (m-1)(m+4)\rho^2 + (m+2)) \frac{\sigma_2^2}{m^2},$$

$$\mu'(0, 2, 4; \rho) = ((m-1)(m-3)\rho^4 + 6(m-1)\rho^2 + 3) \frac{\sigma_2^4}{m^2},$$

$$\mu'(1, 1, 0; \rho) = (m + 2\rho^2) \frac{\sigma_1^2 \sigma_2^2}{m},$$

$$\mu'(1, 0, 0; \rho) = \sigma_1^2,$$

$$\mu'(1, 1, 2; \rho) = ((m+1)\rho^2 + 1) \frac{\sigma_1^2 \sigma_2^2}{m},$$

$$\mu'(2, 0, 0; \rho) = \frac{m+2}{m} \sigma_1^4,$$

$$\mu'(2, 0, 2; \rho) = (-2(m-1)\rho^4 + (m^2 + 3m - 4)\rho^2 + (m+2)) \frac{\sigma_1^4}{m^2},$$

$$\mu'(2, 1, 0; \rho) = \frac{m+2}{m^2} ((m+3)\rho^2 + 1) \sigma_1^4 \sigma_2^2,$$

$$\mu'(2, 1, 2; \rho) = \frac{m+2}{m^2} ((m+3)\rho^2 + 1) \sigma_1^4 \sigma_2^2,$$

$$\mu'(2, 2, 0; \rho) = \frac{m+2}{m^3} (8\rho^4 + 8(m+2)\rho^2 + m(m+2)) \sigma_1^4 \sigma_2^4,$$

$$\mu'(2, 2, 2; \rho) = \frac{m+2}{m^3} (4(m+3)\rho^4 - (m^2 + 7m + 4)\rho^2 + m) \sigma_1^4 \sigma_2^4.$$

Corollary 3.1 The sample product moment correlation between S_1^2 and S_2^2 is given by ρ^2 .

Proof. From the above we have,

$$E(S_1^2) = \mu'(1, 0, 0) = \sigma_1^2,$$

$$E(S_2^2) = \mu'(0, 1, 0) = \sigma_2^2,$$

$$E(S_1^4) = \mu'(2, 0, 0) = \frac{m+2}{m} \sigma_1^4,$$

$$E(S_2^4) = \mu'(0, 2, 0) = \frac{m+2}{m} \sigma_2^4,$$

$$E(S_1^2 S_2^2) = \mu'(1, 1, 0) = \frac{m+2\rho^2}{m} (\sigma_1 \sigma_2)^2.$$

The variances of S_1^2 and S_2^2 are given by

$$V(S_1^2) = \frac{2\sigma_1^4}{m} \text{ and } V(S_2^2) = \frac{2\sigma_2^4}{m}$$

respectively, while the covariance between them is given by

$$\text{Cov}(S_1^2, S_2^2) = E(S_1^2 S_2^2) - E(S_1^2)E(S_2^2) = \frac{2\rho}{m}.$$

Hence, the product moment correlation coefficient between S_1^2 and S_2^2 is given by

$$\text{Corr}(S_1^2, S_2^2) = \frac{\text{Cov}(S_1^2, S_2^2)}{[V(S_1^2) V(S_2^2)]^{1/2}} = \rho^2.$$

Since the correlation between X_1 and X_2 is ρ , the above result is intuitively appealing. Note that if $\rho = 0$, the pdf in (2.2) becomes the product of that of S_1^2 , S_2^2 and R .

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References

- [1] Anderson, T. W. (2003). An Introduction to Multivariate Statistical Analysis, John Wiley and Sons, New York.
- [2] de Waal and Nel, D. J. (1973). On some expectations with respect to Wishart matrices, *South African Statistics Journal*, **7**, 61-67.
- [3] Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population, *Biometrika*, **10**, 507-521.
- [4] Ghosh, B. K. (1966). Asymptotic expansions for the moments of the distribution of correlation coefficient, *Biometrika*, **53**, 258-262.
- [5] Gupta, A. K. and Nagar, D. K. (2000). Matrix Variate Distributions, Chapman and Hall, New York.
- [6] Johnson, N. L.; Kotz, S. and Balakrishnan, N. (1995). Continuous Univariate Distributions Volume 2, John Wiley and Sons, New York.
- [7] Muirhead, R. J. (1982). Aspects of Multivariate Statistical Theory, John Wiley and Sons, New York.
- [8] Wishart, J. (1928). The generalized product moment distribution in samples from a normal multivariate population, *Biometrika*, **A20**, 32-52.